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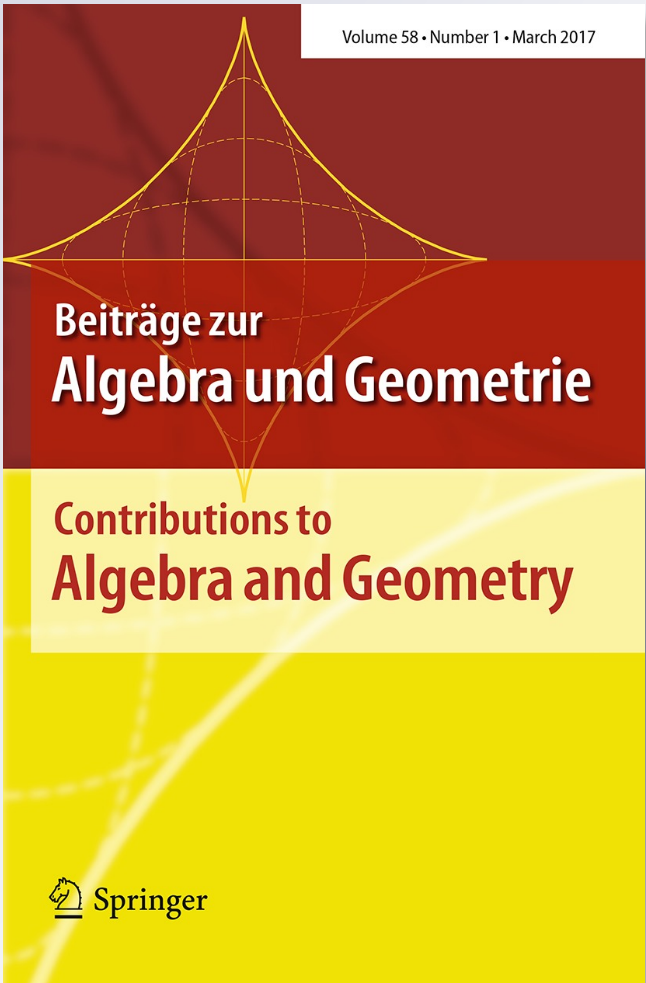

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ORIGINAL PAPER

Conformal geometry of marginally trapped surfaces in $\mathbb{S}^4_1$

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Abstract A spacelike surface S immersed in $\mathbb{S}^4_1$ is marginally trapped if its mean curvature vector is everywhere lightlike. On any oriented spacelike surface S immersed in $\mathbb{S}^4_1$ we show that a choice of orientation of the normal bundle $\nu(S)$ determines a smooth map $G : S \to \mathbb{S}^3$ which we call the *null Gauss map* of S. If S is marginally trapped we show that G is a conformal immersion away from the zeros of certain quadratic Hopf-differential of S and so the surface $G(S)$ is uniquely determined up to conformal transformations of $\mathbb{S}^3$ by two invariants: the normal Hopf differential κ and the schwartzian derivative s. These invariants plus an additional quadratic differential δ are related by a differential equation and determine the geometry of S up to ambient isometries of $\mathbb{S}^4_1$. This allows us to obtain a characterization of marginally trapped surfaces S whose null Gauss image is a *constrained Willmore* surface in $\mathbb{S}^3$ in the sense of Bohle et al. (Calc Var Partial Differ Equ 32:263–277, 2008). As an application of these results we construct and study integrable non-trivial one-parameter deformations of marginally trapped surfaces with non-zero parallel mean curvature vector and those with flat normal bundle.

Keywords Marginally trapped surfaces · Null Gauss map · Conformal invariants · Harmonic map · Integrable deformations · Schwartzian · Associated families

Mathematics Subject Classification 53C42 · 53C50 · 53C43

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1 Introduction

A spacelike surface immersed in a 4-dimensional Lorentz manifold is called marginally trapped if its mean curvature vector is everywhere null or lightlike. The notion of marginally trapped surfaces was introduced by Penrose and plays a key role in the singularity theory of Einstein's equations (Chen 2009). The marginally trapped equation $\langle \vec{H}, \vec{H} \rangle = 0$ is interpreted in relativity theory as the condition describing the event horizon of a black hole (Chen 2009). From a differential geometric point of view marginally trapped surfaces are considered as natural generalizations of minimal surfaces. Different aspects of the geometry of marginally trapped surfaces have drawn the attention of geometers recently. For instance in Aledo et al. (2005) and Huili Liu (2013) the authors provide different Weierstrass-type representation formulas of marginally trapped surfaces in $\mathbb{R}^4_1$. Also in Chen and Van der Veken (2010) the authors classified marginally trapped surfaces with parallel mean curvature in lorentzian spaceforms. The notion of marginally trappedness has also been considered recently in higher dimensions and co-dimensions with very interesting results, see Anciaux (2013) and Anciaux and Godoy (2012).

Our purpose in this article is to study geometric properties of oriented marginally trapped surfaces conformally immersed in $\mathbb{S}^4_1$ using complex analysis, including also the construction of a suitable *null Gauss maps* for such surfaces. We compute the conformal invariants of these maps to obtain results on the congruence of marginally trapped surfaces in $\mathbb{S}^4_1$. Also we characterize oriented marginally trapped surfaces with parallel mean curvature vector and with flat normal bundle in terms of their conformal invariants. As applications of these results we construct spectral non-trivial integrable deformations and describe associated families of marginally trapped surfaces with special properties.

More specifically, given an oriented spacelike surface S immersed in $\mathbb{S}^4_1$ its normal bundle $\nu(S)$ is lorentzian, thus at each point x of S there are two linearly independent null directions say, $n_+(x), n_-(x)$ which vary smoothly with x and determine a pair of smooth maps from S to the 3-sphere $\mathbb{S}^3$, viewed as the manifold of null directions of Minkowski space $\mathbb{R}^5_1$. These maps can be interpreted as (pseudo) inverses of the conformal Gauss map Y introduced by Bryant (1984). We show that a choice of orientation on $\nu(S)$ uniquely determines a distinguished map, say n_+, which we call the *null Gauss map* G of the spacelike surface S. When S is marginally trapped and certain Hopf quadratic differential is never zero on S, then G is a conformal immersion of the surface S with values in the sphere $\mathbb{S}^3$ and its geometry is described by two conformal invariants: the schwartzian s and the normal Hopf differential κ (Burstall et al. 2002; Ma 2005).

The paper is organized as follows: in Sect. 2 we derive the structure equations of spacelike surfaces in $\mathbb{S}^4_1$. Section 3 contains a short survey of $O(3, 1)$-invariant geometry of surfaces in the conformal sphere $\mathbb{S}^3$.

In Sect. 4 we define null Gauss maps and obtain a differential equation (41) relating the conformal invariants s, κ with the δ-quadratic differential, a new geometric invariant of the corresponding marginally trapped surface. A first consequence of this equation is Theorem 4.1, a congruence result, which states that a marginally trapped surface is essentially determined up to ambient isometries by two conformal invariants: κ, s of its null Gauss map. This contrasts for instance with results of Ganchev and Milousheva (2012) in which the authors proved that a marginally trapped surface in Minkowski 4-space is uniquely determined up to ambient lorentzian isometries by seven invariants. Also in Hulett (2005) a congruence theorem for the so-called superconformal spacelike surfaces with zero mean curvature vector in $\mathbb{S}_1^{2n}$ was obtained using a different technique.

Another consequence of Eq. (41) is Theorem 4.2 which states that the null Gauss map of an oriented marginally trapped surface S immersed $\mathbb{S}_1^4$ is a constrained Willmore surface in $\mathbb{S}^3$ if and only if S has non-zero parallel mean curvature vector. Constrained Willmore surfaces in $\mathbb{R}^3$ and $\mathbb{S}^3$ were introduced and studied by Bohle et al. (2008). They are defined as extremes of the Willmore energy with respect to variations preserving the underlying conformal structure of the surface.

In Sect. 5 we consider marginally trapped surfaces admitting non-trivial integrable one parameter deformations. Recall that in classical minimal surface theory in $\mathbb{R}^3$ an interesting problem is to determine whether a given minimal surface can be deformed in a nontrivial way. The oldest known example is the one-parameter deformation of the catenoid into the helicoid (Spivak 1999). In our case the deformation is induced by spectral symmetries of the compatibility equations which give rise to integrable one-parameter deformations of surfaces called associated families. We study here one parameter deformations of two kinds of marginally trapped surfaces in $\mathbb{S}_1^4$ namely, surfaces with non-zero parallel mean curvature vector, and surfaces with flat normal bundle. In the first case the deformation originates in the spectral symmetry of the usual (harmonic) Gauss map with values in the pseudo-riemannian Grasmannian $G_2(\mathbb{R}_1^5)$ of all oriented spacelike 2-planes through the origin of $\mathbb{R}_1^5$. For marginally trapped surfaces in $\mathbb{S}_1^4$ with flat normal bundle we obtain a one-parameter deformation which originates in the so-called *Calapso-Bianchi* or T-transformation of isothermic surfaces in $\mathbb{S}^3$ (Burstall et al. 2002; Ma 2005). We show that both deformations may be unified into an extended action of $\mathbb{C} - \{0\}$ on the class of marginally trapped surfaces with non-zero parallel mean curvature. Finally we give a description of this extended action for non-isotropic marginally trapped tori with non-zero parallel mean curvature vector.

2 Preliminaries

Denote by $\mathbb{R}_1^5$ the real 5-dimensional vector space with canonical coordinates $(x_0, x_1, x_2, x_3, x_4)$ equipped with the Lorentz inner product

$$\langle x, y\rangle = x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_4. \tag{1}$$

We denote by $\{e_0, e_1, \ldots e_4\}$ the canonical basis of $\mathbb{R}_1^5$, where

$$\begin{aligned} e_0 &= (1, 0, 0, 0, 0)^t, \\ e_1 &= (0, 1, 0, 0, 0)^t, \\ &\vdots \\ e_4 &= (0, 0, 0, 0, 1)^t. \end{aligned}$$

The canonical basis is orthonormal with respect to the Lorentz inner product (1):

$$\langle e_i, e_i \rangle = 1, \quad \text{for} \quad 0 \leq i, \leq 3, \quad \langle e_4, e_4 \rangle = -1,$$
$$\langle e_i, e_j \rangle = 0, \quad \text{for} \quad i \neq j.$$

De Sitter 4-space-time is defined as the unit sphere in $\mathbb{R}^5_1$: $\mathbb{S}^4_1 = \{x \in \mathbb{R}^5_1 : \langle x, x \rangle = 1\}$. It is a connected simply connected 4-dimensional manifold which inherits from $\mathbb{R}^5_1$ a lorentzian metric $\langle ., . \rangle$ of constant sectional curvature $+1$. We will consider the complex bilinear extension of the Lorentz metric to $\mathbb{C}^5$ given by $\langle z, w \rangle = z_0 w_0 + z_1 w_1 + z_2 w_2 + z_3 w_3 - z_4 w_4$. We denote by $\mathbb{C}^5_1$ the complex space $\mathbb{C}^5$ endowed with the (pseudo) hermitian inner product $\langle z, \bar{w} \rangle$.
The Lie group $SO(4, 1)$ acts transitively on $\mathbb{S}^4_1$ by isometries, so that choosing $e_0 \in \mathbb{S}^4_1$ as the base point, then $\mathbb{S}^4_1$ is isometric to the (pseudo) riemmanian symmetric space $SO(4, 1)/SO(3, 1)$.
A non-zero vector $X \in \mathbb{R}^5_1$ is said to be *future pointing* if $\langle X, e_4 \rangle < 0$. This determines a time orientation on $\mathbb{S}^4_1$: a non-zero tangent vector $X \in T_p\mathbb{S}^4_1$ is future pointing if its translated to the origin is future pointing. If X is future pointing and satisfies $\langle X, X \rangle = -1$, then (its translated) X lies in the real 4-hyperbolic space $\mathbb{H}^4 = \{x \in \mathbb{R}^5_1 : \langle x, x \rangle = -1, x_4 > 0\}$.

Let Σ be a connected orientable surface and $f : \Sigma \to \mathbb{S}^4_1$ a spacelike immersion, that is, the induced metric $g = f^*\langle ., . \rangle$ is riemannian and it determines a conformal structure on Σ. Then f preserves this conformal structure i.e. $\langle f_z, f_z \rangle = 0$, for every local complex coordinate $z = x + iy$ on Σ, where $\partial_z := \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$, and $\partial_{\bar{z}} := \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$, are the complex partial operators. Equivalently, f is conformal if and only if for any complex coordinate $z = x + iy$,

$$\langle f_x, f_y \rangle = 0, \quad \|f_x\|^2 = \|f_y\|^2 > 0. \tag{2}$$

Conversely, if $f : \Sigma \to \mathbb{S}^4_1$ is a conformal immersion from a Riemann surface, then $\langle f_x, f_y \rangle = 0$, and $\|f_x\|^2 = \|f_y\|^2 \neq 0$, for every local complex coordinate $z = x + iy$. Since the ambient $\mathbb{S}^4_1$ is lorentzian, f_x, f_y are orthogonal and have positive squared norm: $\|f_x\|^2 = \|f_y\|^2 > 0$, and so $f : (\Sigma, g) \to \mathbb{S}^4_1$ is a spacelike isometric immersion, where g is the induced metric. With respect to a local complex coordinate $z = x + iy$ we introduce a conformal parameter u by $\langle f_z, f_{\bar{z}} \rangle =: e^{2u}$, so that $g = 2e^{2u}(dx^2 + dy^2)$ is the local expression of the induced metric.

The pullback bundle by f of the tangent bundle of $\mathbb{S}^4_1$ decomposes into the tangent bundle and the normal bundle of the surface: $f^*(T\mathbb{S}^4_1) = T\Sigma \oplus \nu(f)$. Since $f : \Sigma \to \mathbb{S}^4_1$ is conformal (thus spacelike), and Σ is orientable, the normal bundle $\nu(f)$ is an orientable lorentzian vector bundle. Fixing an orientation on $\nu(f)$, let $\{N_1, N_2\} \subset \Gamma(\nu(f))$ be an (ordered) orthonormal frame satisfying

$$\langle N_2, N_2 \rangle = -1, \quad \langle N_1, N_2 \rangle = 0, \quad \langle N_1, N_1 \rangle = 1.$$

If we demand that N_2 be future pointing, then either $\{N_1, N_2\}$ has the same orientation as $\nu(f)$, or $\{-N_1, N_2\}$ has the same orientation as $\nu(f)$. We say that an orthonormal frame $\{N_1, N_2\} \subset \Gamma(\nu(f))$ is *positively oriented*, if $\{N_1, N_2\}$ has the same orientation as $\nu(f)$ and N_2 is (timelike) future pointing. Note that if $\{N_1, N_2\}$ is positively oriented then $\{-N_1, -N_2\}$ has the same orientation as $\nu(f)$, but it is not positively oriented since $-N_2$ points to the past.

The second fundamental form of f is defined by

$$\langle I\!I(X, Y), N\rangle = -\langle df(X), dN(Y)\rangle, \quad N \in \Gamma(\nu(f)), \quad X, Y \in T\Sigma,$$

and the mean curvature vector of f is defined by $\vec{H} := \frac{1}{2} trace I\!I$. Since f is conformal we have $\langle f_z, f_z\rangle = \langle f_{\bar{z}}, f_{\bar{z}}\rangle = 0$, which implies $0 = \partial_z\langle f_z, f_z\rangle = 2\langle f_{zz}, f_z\rangle$, and $0 = \partial_z\langle f_{\bar{z}}, f_{\bar{z}}\rangle = 2\langle f_{\bar{z}z}, f_{\bar{z}}\rangle$, hence $f_{\bar{z}z}$ has no tangential component. Therefore $f_{\bar{z}z}$ decomposes into its f and $\vec{H}$ components by $f_{\bar{z}z} = -e^{2u} f + e^{2u}\vec{H}$.

On the other hand since f is conformal an easy calculation gives $f_{zz} = 2u_z f_z + \xi_1 N_1 + \xi_2 N_2$, where $\xi_1 := \langle f_{zz}, N_1\rangle, \xi_2 := -\langle f_{zz}, N_2\rangle$, and $\{N_1, N_2\} \subset \Gamma(\nu(f))$ is an orthonormal frame along f. In particular the $(2, 0)$-part of $I\!I$ is given by $I\!I(\partial_z, \partial_z) = \xi_1 N_1 + \xi_2 N_2$. Again since f is conformal an easy calculation gives $f_{zz} = 2u_z f_z + \xi_1 N_1 + \xi_2 N_2$, where $\xi_1 := \langle f_{zz}, N_1\rangle, \xi_2 := -\langle f_{zz}, N_2\rangle$. In particular the $(2, 0)$-part of $I\!I$ is given by $I\!I(\partial_z, \partial_z) = \xi_1 N_1 + \xi_2 N_2$. The mean curvature vector si given in terms of the normal frame by $\vec{H} = h_1 N_1 + h_2 N_2$, where $h_1 := \langle\vec{H}, N_1\rangle$ and $h_2 := -\langle\vec{H}, N_2\rangle$. Defining $\sigma := -\langle\partial_z N_1, N_2\rangle$, we arrive at the structure equations of a conformal immersion $f : \Sigma \to \mathbb{S}^4_1$:

$$\begin{aligned} f_{zz} &= 2u_z f_z + \xi_1 N_1 + \xi_2 N_2 \\ f_{\bar{z}z} &= -e^{2u} f + e^{2u}\vec{H}, \\ \partial_z N_1 &= -h_1 f_z - e^{-2u}\xi_1 f_{\bar{z}} + \sigma N_2, \\ \partial_z N_2 &= h_2 f_z + e^{-2u}\xi_2 f_{\bar{z}} + \sigma N_1. \end{aligned} \tag{3}$$

The compatibility among these equations are just Gauss's, Codazzi's and Ricci's equations:

$$\begin{array}{ll} \text{Gauss}, & 2u_{\bar{z}z} = -e^{2u} + e^{-2u}(|\xi_1|^2 - |\xi_2|^2) - e^{2u}\|\vec{H}\|^2, \\ \text{Codazzi}, & e^{2u}(\partial_z h_1 + \sigma h_2) = \partial_{\bar{z}}\xi_1 + \xi_2\bar{\sigma}, \\ & e^{2u}(\partial_z h_2 + \sigma h_1) = \partial_{\bar{z}}\xi_2 + \xi_1\bar{\sigma}, \\ \text{Ricci}, & Im(\sigma_{\bar{z}}) = e^{-2u} Im(\xi_1\bar{\xi}_2). \end{array} \tag{4}$$

A spacelike surface $f : \Sigma \to \mathbb{S}^4_1$ is called *marginally trapped* if its mean curvature vector is null or lightlike: $\langle\vec{H}, \vec{H}\rangle = 0$. If $\vec{H} \neq 0$, then after a change of orientation of the normal bundle (i.e. after a change of sign $N_1 \mapsto -N_1$) if necessary, the marginally trapped condition $\langle\vec{H}, \vec{H}\rangle = h_1^2 - h_2^2 = 0$, reads $h_1 = h_2$, with $h_1 + h_2 \neq 0$, in this case the mean curvature vector satisfies

$$\vec{H} = h(N_1 + N_2), \quad \text{with } h = h_1 = h_2. \tag{5}$$

We call h the *the mean curvature function* of the surface f with respect to the positively oriented lorentzian normal frame $\{N_1, N_2\}$. From the second structure equation $f_{\bar{z}z} = -e^{2u}f + e^{2u}\vec{H}$, it follows that the mean curvature function h satisfies

$$h = e^{-2u}\langle f_{\bar{z}z}, N_1\rangle = -e^{-2u}\langle f_{\bar{z}z}, N_2\rangle. \tag{6}$$

Hence f is marginally trapped if and only if $\langle f_{\bar{z}z}, (N_1 + N_2)\rangle = 0$. In this case the compatibility conditions or equations of Gauss, Codazzi and Ricci above reduce to:

$$\begin{aligned} 2u_{\bar{z}z} &= -e^{2u} + e^{-2u}(|\xi_1|^2 - |\xi_2|^2), \\ e^{-2u}(\xi_{1\bar{z}} + \xi_2\bar{\sigma}) &= (h_z + \sigma h), \\ e^{-2u}(\xi_{2\bar{z}} + \xi_1\bar{\sigma}) &= (h_z + \sigma h), \\ Im(\sigma_{\bar{z}}) &= e^{-2u}Im(\xi_1\bar{\xi}_2). \end{aligned} \tag{7}$$

The gaussian curvature of the induced metric g is given by $K = -\Delta_g u = -2e^{-2u}u_{\bar{z}z}$, where $\Delta_g = 2e^{-2u}\partial_{\bar{z}}\partial_z$, is the Laplace operator of the induced metric g. From Gauss equation (4) we obtain the expression of the gaussian curvature of the induced metric on Σ,

$$K = 1 - e^{-4u}(|\xi_1|^2 - |\xi_2|^2), \tag{8}$$

Denote by $\nabla^\perp$ the covariant derivative on the normal bundle $\nu(f)$, then $\omega := \langle\nabla^\perp N_2, N_1\rangle$ is the corresponding connection one form. Fixed an orientation on the normal bundle $\nu(f)$ the normal curvature is defined by $d\omega = K^\perp dA_g$, where dA_g is the area form of the induced metric g. Thus $\omega = 2Re(\sigma dz)$, and so $d\omega = -4Im(\sigma_{\bar{z}})dx \wedge dy$. From Ricci equation above it follows that $Im(\sigma_{\bar{z}}) = e^{-2u}Im(\xi_1\bar{\xi}_2)$. Since $dA_g = 2e^{2u}dx \wedge dy$, the normal curvature function is given by

$$K^\perp = -e^{-2u}Im(\sigma_{\bar{z}}) = -e^{-2u}Im(\xi_1\bar{\xi}_2). \tag{9}$$

Then the normal bundle is flat if and only if $K^\perp = 0$.

From Codazzi's equation the $\nabla^\perp$ covariant derivative of the mean curvature vector of a conformal immersion $f : \Sigma \to \mathbb{S}^4_1$ is given by

$$\nabla^\perp_{\partial_z}\vec{H} = (\partial_z h_1 + \sigma h_2)N_1 + (\partial_z h_2 + \sigma h_1)N_2. \tag{10}$$

In particular if f is marginally trapped then in a positively oriented orthonormal frame $\{N_1, N_2\} \subset \Gamma(\nu(f))$ the above formula becomes

$$\nabla^\perp_{\partial_z}\vec{H} = (h_z + \sigma h)(N_1 + N_2). \tag{11}$$

Therefore f has parallel mean curvature vector if and only if $h_z + \sigma h = 0$.

3 Surface theory in the conformal sphere $\mathbb{S}^3$

We give here a brief account of Moebius surface geometry in $\mathbb{S}^3$. For detailed proofs and further developements we refer the reader to Burstall et al. (2002), Hertrich Jeromin (2003) and Ma (2005).

The null or light cone in $\mathbb{R}^5_1$ is defined by

$$\mathcal{L} = \{0 \neq x \in \mathbb{R}^5_1 : \langle x, x\rangle = 0\}. \tag{12}$$

The *future light cone* $\mathcal{L}_+ \subset \mathcal{L}$ consists of future pointing vectors x of $\mathcal{L}$. For every x in $\mathbb{S}^3 \subset \mathbb{R}^4$, the point $(x, 1) \in \mathbb{R}^5_1$ lies in the future light cone $\mathcal{L}_+$. We are using here the fact that any vector x in $\mathbb{R}^5_1$ may be uniquely written as an ordered pair (x', t) with x' in $\mathbb{R}^4$ and t in $\mathbb{R}$, thus giving rise to an isomorphism $\mathbb{R}^4 \oplus \mathbb{R} \to \mathbb{R}^5_1$. In particular points on $\mathcal{L}$ are of the form $(x, \pm\|x\|^2)$, with x in $\mathbb{R}^4$. The map $\mathbb{S}^3 \ni x \mapsto [(x, 1)]$ identifies the unit round sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ with the projectivization of the light cone, $P(\mathcal{L}) \subset \mathbb{RP}^5$.

Let $O_+(4, 1)$ be the group of orthogonal transformations of $\mathbb{R}^5_1$ preserving the time orientation. Then each $F \in O_+(4, 1)$ maps null lines to null lines, and hence preserves the light cone $\mathcal{L}$. Moreover it is easy to see that $O_+(4, 1)$ acts transitively on $\mathbb{S}^3$ by $g.[x] = [gx]$. Here $O_+(4, 1)$ is referred to as the group of Moebius transformations of the conformal sphere $\mathbb{S}^3$. Note that the subgroup of $O(4, 1)$ preserving $P(\mathcal{L}_+)$, is precisely $O_+(4, 1)$.

A smooth map into the conformal sphere $\psi : \Sigma \to \mathbb{S}^3 \equiv P(\mathcal{L})$ can be viewed as a null line subbundle Λ of the trivial bundle $\Sigma \times \mathbb{R}^5_1$ via $\psi(x) = \Lambda_x$, $x \in \Sigma$. A (local) lift of ψ is a smooth map $X : U \to \mathcal{L}$ from an open subset $U \subset \Sigma$, such that the null line spanned by $X(x)$ is Λ_x for every $x \in U$. The map ψ is called a *conformal immersion* if every local lift X of ψ is conformal, i.e. $\langle X_z, X_z\rangle = 0$, $\langle X_z, X_{\bar z}\rangle > 0$, for every coordinate z.

Let $V := span\{X, dX, X_{z\bar z}\}$, where X is a conformal lift of ψ. It is easily seen that V is in fact independent on the choice of a local coordinate z and any particular conformal lift X of ψ. So V can be viewed as a vector sub-bundle $V \subset \mathbb{R}^5_1 \times \Sigma$ on which the ambient metric of $\mathbb{R}^5_1$ induces a vector bundle metric of signature $(3, 1)$. Each fiber V_x determines a Moebius invariant 2-sphere $\mathbb{S}^2(x) \equiv P(V_x \cap \mathcal{L}) \subset P(\mathcal{L}) \cong \mathbb{S}^3$. These spheres altogether comprise the so-called *mean curvature sphere* or *central sphere congruence* of the surface ψ (Burstall et al. 2002). With respect to a fixed a local coordinate $z : U \to \mathbb{C}$ there is a distinguished local lift $Y : U \to \mathcal{L}_+$ of ψ taking values in the future light cone such that

$$\langle Y_z, Y_{\bar z}\rangle = \frac{1}{2},$$

or equivalently $|dY|^2 = |dz|^2$ on U. It is called the *canonical lift* of the surface ψ and is Moebius invariant.

The complementary orthogonal line sub-bundle $V^\perp$ is determined by $\Sigma \times \mathbb{R}^5_1 = V \overset{\perp}{\oplus} V^\perp$ and the connection D on $V^\perp$ is just orthogonal projection of the usual derivative in $\mathbb{R}^5_1$:

$$D_X v = [d_X v]^{\perp}, \quad v \in \Gamma(V^{\perp}), \quad X \in T\Sigma.$$

Let $N \in \Gamma(V)$ be the unique section satisfying

$$\langle N, N\rangle = \langle N, Y_z\rangle = \langle N, Y_{\bar{z}}\rangle = 0, \quad \langle Y, N\rangle = -1.$$

Thus $V = span\{Y, Re(Y_z), Im(Y_z), N\}$ and it is shown in Burstall et al. (2002) that the Moebius invariant frame $\{Y, Y_z, Y_{\bar{z}}, N\} \subset \Gamma(V \otimes \mathbb{C})$ satisfies orthogonally relations given by

$$\begin{aligned} &\langle Y, Y\rangle = \langle N, N\rangle = 0, \quad \langle N, Y\rangle = -1, \\ &\langle Y, dY\rangle = \langle N, dY\rangle = \langle dN, N\rangle = 0, \\ &\langle Y_z, Y_z\rangle = \langle Y_{\bar{z}}, Y_{\bar{z}}\rangle = 0, \quad \langle Y_z, Y_{\bar{z}}\rangle = \frac{1}{2}. \end{aligned} \tag{13}$$

From the above equations it follows that Y_{zz} is orthogonal to Y, Y_z and $Y_{\bar{z}}$ and so there is a unique choice of a local complex function s on Σ for which $Y_{zz} + \frac{s}{2}Y$ is a section of the normal bundle $V^{\perp} \otimes \mathbb{C}$ namely, $\frac{s}{2} = \langle Y_{zz}, N\rangle$. In this way one arrives at the following equation:

$$Y_{zz} + \frac{s}{2}Y = \kappa, \tag{14}$$

defining uniquely the complex valued function s and the section κ of $V^{\perp} \otimes \mathbb{C}$, with respect to the local coordinate z. The function s is interpreted as the *schwartzian derivative* of the conformal immersion ψ, and κ is identified with the *normal valued Hopf differential* of ψ, with respect to the coordinate z. By construction s and κ are Moebius invariants of the immersion ψ with respect to a given coordinate z.

In Burstall et al. (2002) there is an interpretation of κ in terms of euclidean invariants of the immersion ψ which we briefly describe: There is a unique conformal immersion $\widehat{\psi} : \Sigma \to \mathbb{S}^3 \subset \mathbb{R}^4$ satisfying $[(\widehat{\psi}(x), 1)] = \psi(x)$, $\forall x \in \Sigma$. Thus $\phi = (\widehat{\psi}(x), 1)$ is a lift of ψ, which is called the euclidean lift of ψ (Burstall et al. 2002). Let $\nu(\widehat{\psi})$ denote the normal bundle of the immersed surface $\widehat{\psi}$. Then there is a bundle isomorphism $\nu(\widehat{\psi}) \cong V^{\perp}$ given by

$$v \mapsto \langle v, \widehat{H}\rangle(\widehat{\psi}, 1) + (v, 0). \tag{15}$$

where $\widehat{H}$ is the mean curvature vector of $\widehat{\psi}$. Under this isomorphism $\kappa \in \Gamma(V^{\perp} \otimes \mathbb{C})$ corresponds to a complex section $\widehat{\kappa} \in \nu(\widehat{\psi}) \otimes \mathbb{C}$ satisfying $\kappa = \langle\widehat{\kappa}, \widehat{H}\rangle(\widehat{\psi}, 1) + (\widehat{\kappa}, 0)$. Using (14) it is shown that

$$\widehat{\kappa}\frac{dz^2}{|dz|} = \frac{I\!I^{(2,0)}}{|d\phi|},$$

where $I\!I^{(2,0)}$ is the $(2, 0)$-part of the normal bundle valued (euclidean) second fundamental form of $\widehat{\psi}$. In this way κ, up to the isomorphism (15), is the trace free part of the second fundamental form, i.e., the normal bundle valued Hopf differential of $\widehat{\psi}$, scaled by the square root of the $\widehat{\psi}$-induced metric.

From the orthogonality conditions (13) the structural equations of a conformal immersion $\psi : \Sigma \to \mathbb{S}^3$ were obtained in Burstall et al. (2002):

$$\begin{aligned}
&\text{(i) } Y_{zz} = -\frac{s}{2}Y + \kappa,\\
&\text{(ii) } Y_{\bar{z}z} = -\langle \kappa, \bar{\kappa}\rangle Y + \frac{1}{2}N,\\
&\text{(iii) } N_z = -2\langle \kappa, \bar{\kappa}\rangle Y_z - sY_{\bar{z}} + 2D_{\bar{z}}\kappa.
\end{aligned} \tag{16}$$

The compatibility among these equations are the following equations:

$$\begin{array}{ll}
\text{ConformalGauss}: & \frac{s_{\bar{z}}}{2} = 3\langle \bar{\kappa}_z, \kappa\rangle + \langle \bar{\kappa}, \kappa_z\rangle,\\
\text{ConformalCodazzi}: & Im\left(\kappa_{\bar{z}\bar{z}} + \frac{\bar{s}}{2}\kappa\right) = 0.
\end{array} \tag{17}$$

When the local coordinate changes from z to w the new invariants s' and κ' change according to

$$\begin{aligned}
\kappa' &= \kappa \left(\frac{\partial z}{\partial w}\right)^{\frac{3}{2}} \left(\frac{\partial \bar{z}}{\partial \bar{w}}\right)^{-\frac{1}{2}},\\
s' &= s\left(\frac{\partial z}{\partial w}\right)^2 + S_w(z),
\end{aligned} \tag{18}$$

where the usual schwartzian derivative of a meromorphic function $g : \Sigma \to \mathbb{C}$ is given by $S_z(g) = (\frac{g''}{g'})' - \frac{1}{2}(\frac{g''}{g'})^2$. The importance of the conformal Gauss and Codazzi's equations is reflected in the following fundamental theorem of conformal surface theory,

Theorem 3.1 Burstall et al. (2002) *Let Σ be a Riemann surface and $\psi_j : \Sigma \to \mathbb{S}^3$ be conformal immersed surfaces inducing the same Hopf differentials and the same schwartzians. Then there is a Moebius transformation $T : \mathbb{S}^3 \to \mathbb{S}^3$ with $T\psi_1 = \psi_2$.*

Conversely, let κ and s be given data on Σ transforming according (18)*, which also satisfy the conformal Gauss and Codazzi equations* (17)*. Then there exists a conformal immersion $x : \Sigma \to \mathbb{S}^3$ with Hopf differential κ and schwartzian s.*

Remark 3.1 It is proved in Burstall et al. (2002) that $\kappa \frac{dz^2}{|dz|}$ is a globally defined quadratic differential with values in $L \otimes \mathbb{C}$, where L is the real line bundle $(K \otimes \bar{K})^{1/2}$ of densities of conformal weight 1 over Σ (Calderbank 1998). Then for any local coordinate system (U, z), κ is can be viewed just as a local complex function on $U \subset \Sigma$ which transforms according (18).

Remark 3.2 If a conformal immersion $\psi : \Sigma \to \mathbb{S}^3$ has $\kappa \equiv 0$, then the image of ψ is contained in a fixed 2-sphere $\mathbb{S}^2 \subset \mathbb{S}^3$, as follows from (16). Considering ψ as a conformal map $\psi : \Sigma \to \mathbb{S}^2 \equiv \mathbb{CP}^1$, it is shown in Burstall et al. (2002) that $s = (\frac{\psi''}{\psi'})' - \frac{1}{2}(\frac{\psi''}{\psi'})^2$ which is the usual schwartzian derivative of ψ. In this case it is shown that proved that s uniquely determines ψ up to transformations of $PSl(2, \mathbb{C})$, the Moebius transformation group of $\mathbb{CP}^1$.

The map $\gamma : \Sigma \ni x \mapsto V(x)$ with values in the grassmannian $G_{3,1}(\mathbb{R}^5_1)$ is called *the conformal Gauss map* of the immersion ψ (Bryant 1984; Burstall et al. 2002; Ejiri 1988; Palmer 1991; Ma 2005). γ induces a positive definite conformal metric on Σ given by $g_\gamma = \frac{1}{4}\langle d\gamma, d\gamma\rangle = |\kappa|^2|dz|^2$ (Ma 2005). The Willmore energy of the conformal immersion ψ is defined as the total area of (Σ, g_γ) and is given by

$$W(\psi) = \frac{i}{2}\int_\Sigma |\kappa|^2 dz \wedge d\bar{z}, \tag{19}$$

which coincides (up to a constant multiple) with the Willmore energy of the immersion ψ (Burstall et al. 2002). A conformal immersion $\psi : \Sigma \to \mathbb{S}^3$ is called a *Willmore surface* if it extremizes the Willmore energy functional (19). It is known (Burstall et al. 2002) that ψ is Willmore if and only if its conformal invariants κ and s (the Hopf differential and the schwartzian derivative) satisfy the following stronger version of the conformal Codazzi's equation:

$$\kappa_{\bar{z}\bar{z}} + \frac{\bar{s}}{2}\kappa = 0. \tag{20}$$

Spacelike surfaces in $\mathbb{S}^4_1$ are related to surfaces in $\mathbb{S}^3$ and $\mathbb{R}^3$ through a double cover of the conformal Gauss map γ, which is the Bryant's Gauss conformal map Y: given an oriented surface $\psi : \Sigma \to \mathbb{S}^3$ with mean curvature H, the conformal Gauss map Y_ψ assigns to a point $x \in \Sigma$ the oriented sphere $S(x) \subset \mathbb{S}^3$ of radius $|H(x)|^{-1}$ in contact with the surface at $\psi(x)$. Thus Y_ψ takes values in the manifold of all oriented 2-spheres (and planes) in $\mathbb{S}^3$ which is identified with De Sitter 4-space $\mathbb{S}^4_1$ (Hertrich Jeromin 2003). It was already implicit in the work of Blaschke that Y_ψ is marginally trapped (Blaschke 1929; Palmer 1991).

4 The null Gauss map and its conformal invariants

Let $f : \Sigma \to \mathbb{S}^4_1$ be a conformal (spacelike) immersion and fix an orientation on the normal bundle $\nu(f)$. Let $\{N_1, N_2\} \subset \Gamma(\nu(f))$ be a positively oriented lorentzian orthonormal frame. Then for each $p \in \Sigma$ the frame $\{N_1, N_2\}$ determines the null line $span\{N_1(p) + N_2(p)\}$. We claim that this null line depends only on p and not on $\{N_1, N_2\}$. In fact, if $\{N_1', N_2'\} \subset \Gamma(\nu(f))$ is another positively oriented orthonormal frame then both frames are related by a gauge,

$$N_1' = \cosh(s)N_1 + \sinh(s)N_2,$$
$$N_2' = \sinh(s)N_1 + \cosh(s)N_2.$$

from these equations it follows that $N_1' + N_2' = e^s(N_1 + N_2)$, and so $N_1' + N_2'$ and $N_1 + N_2$ generate the same null line. Let

$$G : \Sigma \to \mathbb{S}^3,\ G(x) = [N_1(x) + N_2(x)], \quad x \in \Sigma. \tag{21}$$

i.e. $G(x)$ is the null line generated by $N_1(x) + N_2(x)$, where $\{N_1, N_2\} \subset \Gamma(\nu(f))$ is any positively oriented orthonormal frame. Thus G is well defined by our previous observation and we call it *the null Gauss map of* f.

Denote by $\widehat{G}$ the unique smooth map from Σ to the round euclidean sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ such that $[(\widehat{G}(x), 1)] = G(x)$ for every $x \in \Sigma$, then $\phi =: (\widehat{G}, 1) : \Sigma \to \mathcal{L}_+$ is called *the euclidean lift* of G. Thus $\phi =: (\widehat{G}, 1)$ takes values in the conic section $\mathcal{S} = \{x \in \mathcal{L} : \langle x, e_4\rangle = -1\}$ which inherits from the ambient $\mathbb{R}^5_1$ a positive definite metric of constant curvature $+1$, and so it is a copy of the round 3-sphere of radius one.

For any positively oriented orthonormal frame $\{N_1, N_2\} \subset \Gamma(\nu(f))$, $X = N_1 + N_2$ is a local lift of G with values in $\mathcal{L}_+$. Using the structure equations (3) we see that

$$X_z = N_{1,z} + N_{2,z} = (h_2 - h_1) f_z + e^{-2u}(\xi_2 - \xi_1) f_{\bar{z}} + \sigma X. \tag{22}$$

Hence $\langle X, f_z\rangle = \langle X, f_{\bar{z}}\rangle = 0$. Moreover since $\langle f_z, f_z\rangle = \langle f_{\bar{z}}, f_{\bar{z}}\rangle = 0$, $\langle f_z, f_{\bar{z}}\rangle = e^{2u}$, then

$$\langle X_z, X_z\rangle = \langle N_{1,z} + N_{2,z}, N_{1,z} + N_{2,z}\rangle = (h_2 - h_1)(\xi_2 - \xi_1) \tag{23}$$

Thus if f is marginally trapped $\langle X_z, X_z\rangle = 0$. Let Z be another local lift of G, then $X = \lambda Z$ for some smooth non-zero function λ. With respect to a local coordinate z we compute $X_z = \lambda_z Z + \lambda Z_z$. Since $\langle Z, Z\rangle = 0$, then $0 = \langle Z, Z_z\rangle$, hence $0 = \langle X_z, X_z\rangle = \lambda^2\langle Z_z, Z_z\rangle$, from which $\langle Z_z, Z_z\rangle = 0$ follows. On the other hand since $X_{\bar{z}} = \lambda_{\bar{z}} Z + \lambda Z_{\bar{z}}$, then from $\langle Z, Z_{\bar{z}}\rangle = 0$, and (22) we obtain

$$\lambda^2\langle Z_z, Z_{\bar{z}}\rangle = e^{-2u}|\xi_1 - \xi_2|^2 = \langle X_z, X_{\bar{z}}\rangle. \tag{24}$$

Hence away from the zeros of $\xi_1 - \xi_2$ it follows that $\langle X_z, X_{\bar{z}}\rangle > 0$ and $\langle Z_z, Z_{\bar{z}}\rangle > 0$. In particular if $\xi_1 - \xi_2$ is never zero on Σ then $G : \Sigma \to \mathbb{S}^3$ is a conformal immersion.

We call $q := (\xi_1 - \xi_2)dz^2$ *the Hopf quadratic differential* of the marginally trapped surface $f : \Sigma \to \mathbb{S}^4_1$. The quadratic Hopf differential was introduced in Aledo et al. (2005) for marginally trapped surfaces in $\mathbb{R}^4_1$. We have proved the following Lemma:

Lemma 4.1 *Let $f : \Sigma \to \mathbb{S}^4_1$ be a conformally immersed marginally trapped surface and q its quadratic Hopf differential. Then every (local) lift Z of the null Gauss map G satisfies $\langle Z_z, Z_z\rangle = 0$ and $\langle Z_z, Z_{\bar{z}}\rangle > 0$ away from the zeros of q. In particular if $q(x) \neq 0, \forall x \in \Sigma$ then $G : \Sigma \to \mathbb{S}^3$ is a conformal immersion.*

Since $(\widehat{G}, 1)$ is a lift of G, then away from the zeros of q, $\widehat{G}$ satisfies $\langle \widehat{G}_z, \widehat{G}_z\rangle = 0$ and $\langle \widehat{G}_z, \widehat{G}_{\bar{z}}\rangle > 0$, where $\langle ., .\rangle$ is the round metric on the sphere $\mathbb{S}^3$. Thus if q is never zero $\widehat{G}$ is a conformal immersion into the round 3-sphere.

Let $f : \Sigma \to \mathbb{S}^4_1$ be a spacelike immersion then from Ricci's equation $\nu(f)$ is flat if and only if $Im(\sigma_{\bar{z}}) = 0$. In this case $\sigma_{\bar{z}} - \overline{\sigma_{\bar{z}}} = \sigma_{\bar{z}} - \overline{\sigma}_z = 0$ which shows that the real one form $\eta := -\sigma dz - \overline{\sigma} d\bar{z}$ is closed. Hence there is a locally defined smooth real function β such that $d\beta = \eta$. One can define a new positively oriented orthonormal lorentzian frame $\{N'_1, N'_2\}$ by

$$N_1' = \cosh(\beta)N_1 + \sinh(\beta)N_2, \quad N_2' = \sinh(\beta)N_1 + \cosh(\beta)N_2.$$

Then it is easy to check that the new frame $\{N_1', N_2'\}$ has structure function $\sigma' = 0$, so that $\{N_1', N_2'\}$ is a $\nabla^\perp$-parallel frame which is unique up to (constant) hyperbolic rotations in $\nu(f)$. We keep denoting by $\{N_1, N_2\}$ this new positively oriented $\nabla^\perp$-parallel orthonormal frame. If f is marginally trapped then Codazzi's equations (4) reduce to

$$\xi_{1,\bar{z}} = \xi_{2,\bar{z}} = e^{2u}h_z, \quad h = h_1 = h_2, \tag{25}$$

which imply $(\xi_1 - \xi_2)_{\bar{z}} = e^{2u}(h - h)_z = 0$, hence q is holomorphic. Conversely, if q is holomorphic then again by Codazzi's equation we obtain $0 = (\xi_1 - \xi_2)_{\bar{z}} = \bar{\sigma}(\xi_1 - \xi_2)$. If q does not vanish identically then σ must be zero away from the isolated zeros of q, thus $\sigma \equiv 0$ by continuity. We have proved the following,

Lemma 4.2 *Let $f : \Sigma \to \mathbb{S}^4_1$ be a marginally trapped surface. If f has flat normal bundle the Hopf differential $q = (\xi_2 - \xi_1)dz^2$ is holomorphic. Conversely, if q is holomorpic and non-identically zero, then f has flat normal bundle.*

Remark 4.1 (i) If a conformally immersed surface $f : \Sigma \to \mathbb{S}^4_1$ has zero mean curvature vector then its normal bundle is not necessarily flat. In this case the Hopf differential q is holomorphic as consequence of Codazzi's equations (4).

(ii) If $f : \Sigma \to \mathbb{S}^4_1$ is marginally trapped with parallel mean curvature vector then $\nu(f)$ is flat (Rahim Elghanmi 1996) and so q is holomorphic by Lemma 4.2.

(iii) From (5) the $\nabla^\perp$-derivative of the mean curvature vector of a marginally trapped surface in a positively oriented normal frame is given by (11). If $\nabla^\perp \vec{H} = 0$ then $\nu(f)$ is flat (Rahim Elghanmi 1996), hence the mean curvature function h is constant in a positively oriented $\nabla^\perp$-parallel frame $\{N_1, N_2\} \subset \Gamma(\nu(f))$. Conversely if $\nu(f)$ is flat, then $\sigma = 0$ for any $\nabla^\perp$-parallel orthonormal frame $\{N_1, N_2\} \subset \Gamma(\nu(f))$.

Remark 4.2 If $q \equiv 0$, then by (24) $N_1 + N_2$ is a constant null line for every oriented lorentzian frame $\{N_1, N_2\}$, hence the null Gauss map G is constant. Since $\langle f, N_1 + N_2\rangle = 0$, the surface f has constant curvature $K = 1$ by (8) and lies in the degenerated hypersurface $M_0 \subset \mathbb{S}^4_1$, which is the intersection of the degenerate 4-plane $[N_1+N_2]^\perp$ in $\mathbb{R}^5_1$ with $\mathbb{S}^4_1$. For instance this is just the case of any marginally trapped surface $f : \mathbb{S}^2 \to \mathbb{S}^4_1$ with flat normal bundle. In fact since q is holomorphic on $\mathbb{S}^2$, it must vanish.

4.1 Spacelike isothermic surfaces

The normal valued quadratic Hopf differential of a spacelike immersion $f : \Sigma \to \mathbb{S}^4_1$ is the $\Gamma(\nu(f) \otimes \mathbb{C})$-valued two-form

$$\Omega = \xi_1 N_1 dz^2 + \xi_2 N_2 dz^2,$$

defined in terms of an orthonormal frame $\{N_1, N_2\} \subset \Gamma(\nu(f))$, where ξ_1, ξ_2 are the coefficients of $I\!I(\partial_z, \partial_z)$, the $(2, 0)$-component of the second fundamental form of

f. The spacelike surface $f : \Sigma \to \mathbb{S}^4_1$ is called isothermic (Wang 2012) if for each point $x \in \Sigma$ there is a coordinate z for which the normal valued Hopf differential Ω is real-valued. From Ricci's equation (4) it follows that every isothermic spacelike immersion in $\mathbb{S}^4_1$ has flat normal bundle.

4.2 Non-isotropic spacelike surfaces

A conformally (hence spacelike) immersed surface $f : \Sigma \to \mathbb{S}^4_1$ is called *non-isotropic* if the quartic complex differential $Q = \langle f_{zz}, f_{zz}\rangle dz^4$ is never zero on Σ. The quartic complex differential Q was introduced in Bryant (1984) in the context of the conformal Gauss map. In terms of an orthonormal frame $\{N_1, N_2\} \subset \Gamma(\nu(f))$, $Q = (\xi_1^2 - \xi_2^2)dz^4$, thus if f is non-isotropic then the Hopf differential $q = (\xi_1 - \xi_1)dz^2$ is never zero and so the null Gauss map $G : \Sigma \to \mathbb{S}^3$ is a conformal immersion.

The notion of isotropy has an interpretation in terms of the *curvature hyperbola* which is the image of the unit circle on $T_p\Sigma$ under the second fundamental form of f:

$$\{I\!I_p(X, X) : X \in T_p\Sigma, \|X\|^2 = 1\} \subset T_p^{\perp}\Sigma$$

It is shown that f is non-isotropic if and only if the curvature hyperbola at each point of Σ is non-equilateral (Rahim Elghanmi 1996). A conformal non-isotropic spacelike immersion $f : \Sigma \to \mathbb{S}^4_1$ with zero mean curvature vector is also called harmonic superconformal (Hulett 2005). Hence non-isotropic marginally trapped surfaces can be viewed as natural generalizations of harmonic superconformal surfaces.

4.3 Sphere congruences

Let $f : \Sigma \to \mathbb{S}^4_1$ be a non-isotropic marginally trapped surface with null Gauss map G and consider the central sphere congruence of the surface $G : \Sigma \to \mathbb{S}^3$, given by the subbundle $V = span\{X, dX, X_{z\bar{z}}\} \subset \Sigma \times \mathbb{R}^5_1$, where $X : \Sigma \to \mathcal{L}_+$ is any local lift of G. Since $\mathbb{S}^4_1$ identifies with the manifold of oriented 2-spheres in $\mathbb{S}^3$, the immersion f is associated to the 2-sphere congruence $\Sigma \ni x \mapsto S(x)$, where $S(x)$ is the 2-sphere obtained by projectivization of the intersection of the Minkowski vector subspace $f^{\perp}(x) \subset \mathbb{R}^5_1$ with the null cone $\mathcal{L}$:

$$S(x) = P(f^{\perp}(x) \cap \mathcal{L}) \subset \mathbb{S}^3.$$

Note that the antipodal surface $(-f)$ determines the same sphere congruence $x \mapsto S(x)$. We say that $S(x)$ is oriented if it is associated to f, and opposite oriented if it is associated to $-f$. We claim that $f^{\perp} = V$, i.e. both sphere congruences coincide. To prove the claim we use the local lift of G given by $X := N_1 + N_2 : U \to \mathcal{L}_+$, where $\{N_1, N_2\} \subset \Gamma(\nu(f))$ is a positively oriented orthonormal lorentzian frame. Thus $V = span\{X, Re(X_z), Im(X_z), X_{z\bar{z}}\}$. In particular $\langle X, f\rangle = 0$ since N_1, N_2 are normal to f. On the other hand from (22),

$$X_z = e^{-2u}(\xi_2 - \xi_1) f_{\bar{z}} + \sigma X. \tag{26}$$

Hence $\langle f, X_z\rangle = \langle f, X_{\bar{z}}\rangle = 0$, or $\langle f, dX\rangle = 0$. Since every lift W of G is a multiple of X by some function, then W satisfies $\langle f, W\rangle = 0$ and $\langle f, dW\rangle = 0$. This just says that G is an envelope of the congruence determined by f (Hertrich Jeromin 2003). On the other hand taking $\partial_{\bar{z}}$ on (26) and using again (3) yields

$$X_{z\bar{z}} = e^{-2u}(\xi_2 - \xi_1)(\bar{\xi}_1 N_1 + \bar{\xi}_2 N_2) + \sigma e^{-2u}(\bar{\xi}_2 - \bar{\xi}_1) f_z + (\sigma_{\bar{z}} + |\sigma|^2)X,$$

from which $\langle f, X_{z\bar{z}}\rangle = 0$ follows and so $V \subseteq f^{\perp}$. Thus $V = f^{\perp}$ since V has rank four. We have proved the following

Proposition 4.1 *Let $f : \Sigma \to \mathbb{S}^4_1$ be a non-isotropic conformal marginally trapped immersion with null Gauss map $G : \Sigma \to \mathbb{S}^3$. Then G is an envelope of the spherical congruence determined by f. Moreover, the central sphere congruence of the null Gauss map G coincides with the spherical congruence determined by $\pm f$.*

Recall from Sect. 3 that the correspondence $\gamma_G : x \mapsto V(x)$ defines the *conformal Gauss map* of the surface $G : \Sigma \to \mathbb{S}^3$. Since $V = f^{\perp}$, then γ_G takes values in $G_{3,1}(\mathbb{R}^5_1)$ the grassmannian of all subspaces of $\mathbb{R}^5_1$ with signature $(+++-)$. Since V and $V^{\perp} = \mathbb{R}f$ determine each other then either of them can be used to define the conformal Gauss map of G. Thus for each $x \in \Sigma$, $\gamma_G(x) = \mathbb{R}f(x)$ belongs to the manifold of all spacelike lines through the origin of $\mathbb{R}^5_1$ which identifies also with $G_{3,1}(\mathbb{R}^5_1)$. Note that the projection $\mathbb{S}^4_1 \to G_{3,1}(\mathbb{R}^5_1)$ given by $P : p \mapsto \mathbb{R}p$ is a lorentzian double cover. Intersecting the spacelike line $\gamma_G(x) = \mathbb{R}f(x)$ with $\mathbb{S}^4_1$ we obtain $\{+f(x), -f(x)\} \subset \mathbb{S}^4_1$ which is just the fiber of P over $G(x) \in \mathbb{S}^3$. Thus the surface f and its antipodal $-f$ have the same null Gauss map G. Then the null Gauss map G can be considered as a pseudo-inverse of the conformal Gauss map γ_G.

4.4 A differential equation relating κ, s and δ

Let Y be the canonical lift of G with respect to a local coordinate z. Then there is a non-zero function τ such that $X = \tau Y$. Using (22), we compute

$$\tau_z Y + \tau Y_z = X_z = e^{-2u}(\xi_2 - \xi_1) f_{\bar{z}} + \sigma X. \tag{27}$$

Hence $\langle X_z, X_{\bar{z}}\rangle = \frac{\tau^2}{2} = \tau^2\langle Y_z, Y_{\bar{z}}\rangle = e^{-2u}|\xi_2 - \xi_1|^2$, so that

$$\tau = \sqrt{2}e^{-u}|\xi_2 - \xi_1|. \tag{28}$$

Hence we obtain the canonical lift of G in terms of $X = N_1 + N_2$:

$$Y = \frac{e^u}{\sqrt{2}|\xi_2 - \xi_1|}(N_1 + N_2).$$

A routine computation using the structure equations of f shows that Y is in fact independent on any particular choice of a positively oriented lorentzian frame $\{N_1, N_2\}$. On the other hand

$$\begin{aligned}\tau_{zz}Y + 2\tau_z Y_z + \tau Y_{zz} &= X_{zz}\\ &= (e^{-2u}(\xi_2 - \xi_1))_z f_{\bar{z}} + e^{-2u}(\xi_2 - \xi_1)(-e^{2u} f + e^{2u} hX)\\ &\quad + \sigma_z X + \sigma\{e^{-2u}(\xi_2 - \xi_1) f_{\bar{z}} + \sigma X\}.\end{aligned}$$

Adding and substracting $\tau \frac{s}{2} Y$ we obtain

$$\begin{aligned}(\tau_{zz} - \tau \frac{s}{2})Y + 2\tau_z Y_z + \tau(Y_{zz} + \frac{s}{2}Y) &= (e^{-2u}(\xi_2 - \xi_1))_z f_{\bar{z}}\\ + e^{-2u}(\xi_2 - \xi_1)(-e^{2u} f + e^{2u} hX) &+ \sigma_z X + \sigma\{e^{-2u}(\xi_2 - \xi_1) f_{\bar{z}} + \sigma X\}.\end{aligned} \tag{29}$$

Comparing the $V^{\perp}$ components in this identity we obtain the equality

$$(\xi_1 - \xi_2) f = \tau \left(Y_{zz} + \frac{s}{2} Y\right) = \tau\kappa, \tag{30}$$

Inserting the function τ of (28) we obtain a formula for the normal valued Hopf differential of G which makes sense only if the Hopf quadratic differential of f is non-zero:

$$\kappa = \frac{(\xi_1 - \xi_2)e^u}{\sqrt{2}|\xi_1 - \xi_2|} f. \tag{31}$$

Using the polar form $(\xi_1 - \xi_2) = |\xi_1 - \xi_2| e^{i\theta}$ the above expression becomes $\kappa = \frac{e^{u+i\theta}}{\sqrt{2}} f$ and so by Remark 3.1 we identify

$$\kappa \equiv \frac{e^{u+i\theta}}{\sqrt{2}}, \quad \text{where } \frac{(\xi_1 - \xi_2)}{|\xi_1 - \xi_2|} = e^{i\theta}. \tag{32}$$

In particular we recover the conformal parameter from κ above:

$$e^{2u} = 2\langle \kappa, \overline{\kappa}\rangle. \tag{33}$$

In Burstall et al. (2002) it is shown that any section $v \in \Gamma(V \otimes \mathbb{C})$ can be decomposed as follows:

$$v = -\langle v, N\rangle Y - \langle v, Y\rangle N + 2\langle v, Y_{\bar{z}}\rangle Y_z + 2\langle v, Y_z\rangle Y_{\bar{z}}. \tag{34}$$

We use this formula to expand the particular section $f_z \in \Gamma(V \otimes \mathbb{C})$. Since $\tau Y = N_1 + N_2 = X$, it follows $\langle f_z, Y\rangle = 0$. Also from $0 = \langle f, Y_z\rangle_z = \langle f_z, Y_z\rangle + \langle f, Y_{zz}\rangle$, Eq. (16)-($i$), and $\langle f, Y\rangle = 0$, we compute

$$\langle f_z, Y_z\rangle = -\langle f, Y_{zz}\rangle = -\left\langle f, -\frac{s}{2} Y + \kappa\right\rangle = -\langle f, \kappa\rangle = -\frac{e^{u+i\theta}}{\sqrt{2}}.$$

On the other hand since $0 = \langle f, Y_{\bar z}\rangle_z = \langle f_z, Y_{\bar z}\rangle + \langle f, Y_{z\bar z}\rangle$, then

$$\langle f_z, Y_{\bar z}\rangle = -\langle f, Y_{z\bar z}\rangle = |\kappa|^2\langle Y, f\rangle - \frac{1}{2}\langle N, f\rangle = 0.$$

Also $\langle f, N\rangle = 0$, implies $\langle f_z, N\rangle + \langle f, N_z\rangle = 0$. Hence $\langle f_z, N\rangle = -\langle f, N_z\rangle = -2\langle f, D_{\bar z}\kappa\rangle$. Since $D_{\bar z}\kappa = (u + i\theta)_{\bar z}\kappa$, then

$$\langle f_z, N\rangle = -\sqrt{2}(u + i\theta)_{\bar z}e^{u+i\theta}.$$

From these equations and using (34) with $v = f_z$, we obtain

$$f_z = \sqrt{2}e^{u+i\theta}\{(u + i\theta)_{\bar z}Y - Y_{\bar z}\}. \tag{35}$$

Therefore,

$$f_{z\bar z} = \sqrt{2}e^{u+i\theta}\left\{((u + i\theta)_{\bar z})^2 + (u + i\theta)_{\bar z\bar z} + \frac{\bar s}{2}\right\}Y - \sqrt{2}e^{u+i\theta}\bar\kappa. \tag{36}$$

On the other hand using the structure equations of the immersion f and $X = N_1 + N_2 = \tau Y$, we obtain

$$f_{z\bar z} = -e^{2u}f + e^{2u}hX = -e^{2u}f + e^{2u}h\tau Y. \tag{37}$$

Note that $\sqrt{2}e^{u+i\theta}\bar\kappa = e^{2u}f$, so that equating (36) and (37) gives

$$e^{2u}h\tau = \sqrt{2}e^{u+i\theta}\left\{((u + i\theta)_{\bar z})^2 + (u + i\theta)_{\bar z\bar z} + \frac{\bar s}{2}\right\}.$$

Inserting the function τ given by (28) in this expression we obtain the following formula:

$$h|\xi_2 - \xi_1|e^{-i\theta} = ((u + i\theta)_{\bar z})^2 + (u + i\theta)_{\bar z\bar z} + \frac{\bar s}{2}, \tag{38}$$

or conjugating both sides,

$$h(\xi_1 - \xi_2) = ((u - i\theta)_z)^2 + (u - i\theta)_{zz} + \frac{s}{2}. \tag{39}$$

Now recall the connection D on the normal bundle $V^\perp$. Any section $v \in \Gamma(V^\perp)$ can be written as $v = bf$ for some smooth function b. Thus $d_X(bf) = d_Xbf + bd_Xf$. The condition $df \perp f$ implies $D_Xf = 0$, hence

$$D_X(v) = (d_Xb)f. \tag{40}$$

Thus we may identify $D_X(v) \equiv d_Xb$. Since $\kappa \equiv \frac{e^{u+i\theta}}{\sqrt{2}}$, we compute

$$D_{\bar z}D_{\bar z}\kappa = \kappa_{\bar z\bar z} = \left((u + i\theta)_{\bar z}^2 + (u + i\theta)_{\bar z\bar z}\right)\kappa.$$

On the other hand since $\overline{h(\xi_1 - \xi_2)\kappa} = \frac{e^u}{\sqrt{2}} h|\xi_2 - \xi_1|$, then $\overline{h(\xi_1 - \xi_2)\kappa}$ is real valued and so Eq. (38) becomes

$$\kappa_{\bar{z}\bar{z}} + \frac{\bar{s}}{2}\kappa = Re\left(\overline{h(\xi_1 - \xi_2)\,\kappa}\right). \tag{41}$$

Equation (41) relates the quadratic differential $h(\xi_1 - \xi_2)dz^2$ of a marginally trapped surface $f : \Sigma \to \mathbb{S}^4_1$ to the conformal invariants κ, s of its null Gauss map G. Since the quadratic differential $\delta := h(\xi_1 - \xi_2)dz^2$ plays a key role in (41), we call it the δ-differential of the marginally trapped surface f.

Remark 4.3 Equation (41) implies the conformal Codazzi equation $Im(\kappa_{\bar{z}\bar{z}} + \frac{\bar{s}}{2}\kappa) = 0$. The conformal Gauss equation (17) may be recovered from (39) by a long calculation using Gauss, Codazzi and Ricci's equations (4).

4.5 Congruence

A basic question is to what extent a marginally trapped surface is determined by the conformal invariants of its null Gauss map. We obtain the following:

Theorem 4.1 *Let $f, f' : \Sigma \to \mathbb{S}^4_1$ be non-isotropic marginally trapped surfaces with null Gauss maps G, G'. If $\kappa = \kappa'$, $s = s'$ then there is an isometry Φ of $\mathbb{S}^4_1$ such that $\Phi f = f'$. As a consequence of this $\delta = \delta'$.*

Proof By Theorem 3.1 there is a Moebius transformation $T \in O_+(4,1)$ of $\mathbb{S}^3$ such that $TG = G'$. Recall that the Moebius group $O_+(4,1)$ acts on $\mathbb{S}^3$ by $T([x]) = [Tx]$, $\forall x \in \mathcal{L}$. Let Y be the canonical lift of G with respect to a holomorphic coordinate z, then $Y' = TY$ is the canonical lift of G' with respect to z. Since $V = span\{Y, Re(Y_z), Im(Y_z), Y_{z\bar{z}}\}$, it follows that $TV = V'$ and so $TV^{\perp} = V'^{\perp}$. This last equality implies $Tf = \pm f'$ where the sign ambiguity reflects the fact that the sphere congruences determined by f and f' are (modulo Moebius transformations) equal up to orientation. Defining $\Phi = T$, if $Tf = f'$ and $\Phi = -T$, if $Tf = -f'$, then Φ is an isometry of $\mathbb{S}^4_1$ satisfying $\Phi f = f'$. In particular if T is the identity, then $G = G'$ and so $V^{\perp} = \mathbb{R}f = \mathbb{R}f'$, which implies $f' = \pm f$.

Let $\{N_1, N_2\} \subset \Gamma(\nu(f))$ be a positively oriented orthonormal frame, then $\{\Phi N_1, \Phi N_2\} \subset \Gamma(\nu(f'))$ is an orthonormal frame. We can choose an orientation on $\nu(f')$ so that $\{\Phi N_1, \Phi N_2\} \subset \Gamma(\nu(f'))$ is a positively oriented normal frame along f'. Since $\vec{H} = h(N_1 + N_2)$ is the mean curvature vector of f, then $\Phi\vec{H} = h(\Phi N_1 + \Phi N_2)$ is the mean curvature vector of f'. Also since $I\!I(\partial_z, \partial_z) = \xi_1 N_1 + \xi_2 N_2$, then $I\!I'(\partial_z, \partial_z) = \xi_1 \Phi N_1 + \xi_2 \Phi N_2$ and so $\delta' = h(\xi_1 - \xi_2)dz^2 = \delta$. □

The following result is a partial converse of the previous Theorem:

Lemma 4.3 *Let $f, f' : \Sigma \to \mathbb{S}^4_1$ be non-isotropic marginally trapped surfaces which induce the same conformal metric. If either*

(i) *f, f' are both non-stationary and $\delta = \delta'$, or*

(ii) *f, f' are both stationary with $q = q'$, then there is an isometry Φ of $\mathbb{S}^4_1$ such that $\Phi \circ f = f'$.*

Proof Assume first that f, f' are both non-stationary with $\delta = \delta'$ i.e. $h(\xi_1 - \xi_2)dz^2 = h'(\xi'_1 - \xi'_2)dz^2$, hence $h(\xi_1 - \xi_2) = h'(\xi'_1 - \xi'_2)$. Since h, h' are real and non-zero, we may assume they are both positive (if say $h < 0$, we can replace f by its antipodal $-f$ which has mean curvature function $-h > 0$). Since by hypothesis the Hopf differentials q, q' are never zero, we use the polar form $\xi_1 - \xi_2 = |\xi_1 - \xi_2|e^{i\theta}$ and $\xi'_1 - \xi'_2 = |\xi'_1 - \xi'_2|e^{i\theta'}$. Hence the equality $\delta = \delta'$ implies

$$h|\xi_1 - \xi_2|e^{i\theta} = h'|\xi'_1 - \xi'_2|e^{i\theta'}.$$

It follows that $\theta - \theta' = 2k\pi$ with integer k. Since by hypothesis f and f' induce the same conformal metric, we have $u = u'$ and so (32) implies $\kappa = \kappa'$. On the other hand from $\delta = \delta'$ and (41) it follows that $s = s'$. Thus G, G' have the same conformal invariants κ and s, hence (i) follows by applying the preceding Theorem.

If now f, f' are both stationary with $q = q'$, then $|\xi_1 - \xi_2|e^{i\theta} = |\xi'_1 - \xi'_2|e^{i\theta'}$, and so $\theta - \theta'$ is an integer multiple of 2π. Thus since $u = u'$ by hypothesis, (32) implies $\kappa = \kappa'$. Since f, f' are both stationary, then $\delta = \delta' = 0$. Thus from (41), we conclude that $s = s'$, and so G, G' have the same conformal invariants. □

A conformal immersed surface $\psi : \Sigma \to \mathbb{S}^3$ is called *constrained Willmore* if it extremizes the Willmore energy functional with respect to variations through conformal immersions (Burstall et al. 2002). It has been proved in Bohle et al. (2008) that ψ is constrained Willmore if and only if its conformal invariants κ, s satisfy

$$\kappa_{\bar{z}\bar{z}} + \frac{\bar{s}}{2}\kappa = Re(\bar{\eta}\kappa), \tag{42}$$

for some holomorphic quadratic differential ηdz^2 on Σ. Equations (42) and (41) are related. In fact, we have seen before that for an immersed non-isotropic marginally trapped surface $f : \Sigma \to \mathbb{S}^4_1$ the quantity $\overline{h(\xi_1 - \xi_2)\kappa}$ is real, so that we ask under what conditions is $\delta = h(\xi_1 - \xi_2)dz^2$ holomorphic.

Lemma 4.4 *Let $f : \Sigma \to \mathbb{S}^4_1$ be a non-isotropic conformally immersed marginally trapped surface with non-zero mean curvature vector. Then the following affirmations are equivalent:*

(i) *The quartic complex differential $Q = \langle f_{zz}, f_{zz}\rangle dz^4$ is holomorphic,*
(ii) *The quadratic complex differential $\delta = h(\xi_1 - \xi_2)dz^2$ is holomorphic,*
(iii) *f has parallel mean curvature vector.*

Proof Let $\{N_1, N_2\} \subset \Gamma(\nu(f))$ be a positively oriented orthonormal frame, then the quartic differential becomes $Q = (\xi_1^2 - \xi_2^2)dz^4$, where $\xi_1 = \langle f_{zz}, N_1\rangle, \xi_2 = -\langle f_{zz}, N_2\rangle$ and $\vec{H} = h(N_1 + N_2)$. Since f is marginally trapped Codazzi's equations (4) reduce to

$$e^{-2u}(\xi_{1\bar{z}} + \bar{\sigma}\xi_2) = e^{-2u}(\xi_{2\bar{z}} + \bar{\sigma}\xi_1) = h_z + \sigma h.$$

Using these quations we compute

$$\begin{aligned}(\xi_1^2-\xi_2^2)_{\bar z} &= 2\xi_1\partial_{\bar z}\xi_1 - 2\xi_2\partial_{\bar z}\xi_2\\ &= 2\xi_1(e^{2u}(h_z+\sigma h)-\xi_2\bar\sigma) - 2\xi_2(e^{2u}(h_z+\sigma h)-\xi_1\bar\sigma)\\ &= 2e^{2u}(h_z+\sigma h)(\xi_1-\xi_2).\end{aligned}$$

Since f is non-isotropic q is never zero, thus Q is holomorphic if and only if $h_z+\sigma h = 0$, which is just the parallel mean curvature equation (11). This proves (i) ⇔ (iii).

Again from Codazzi's equation we get $(\xi_1-\xi_2)_{\bar z} = \bar\sigma(\xi_1-\xi_2)$, which implies $(h(\xi_1-\xi_2))_{\bar z} = (h_{\bar z}+\bar\sigma h)(\xi_1-\xi_2)$. Hence $(\xi_1^2-\xi_2^2)_{\bar z} = 2e^{2u}(h(\xi_1-\xi_2))_{\bar z}$, thus δ is holomorphic if and only if Q is holomorphic which proves (i) ⇔ (ii). □

Note for instance that there is no non-isotropic spacelike immersion $f:\mathbb{S}^2\to\mathbb{S}^4_1$ with parallel non-zero mean curvature vector. Isotropic marginally trapped surfaces in $\mathbb{R}^4_1$, and $\mathbb{S}^4_1$ have been considered in Cabrerizo et al. (2010).
Another consequence of Eq. (41) is the following result:

Theorem 4.2 *Let $f:\Sigma\to\mathbb{S}^4_1$ be a non-isotropic conformally immersed marginally trapped surface with null Gauss map G and mean curvature vector $\vec{H}$. Then,*

(a) *$\vec{H}=0$ if and only if $G:\Sigma\to\mathbb{S}^3$ is a Willmore surface.*
(b) *If $\vec{H}\neq 0$, then $\nabla^\perp\vec{H}=0$ if and only if $G:\Sigma\to\mathbb{S}^3$ is a constrained Willmore surface.*

Proof $\vec{H}=0$ if and only if $\delta\equiv 0$ by (5) if and only if Eq. (41) becomes $\kappa_{\bar z\bar z}+\frac{\bar s}{2}\kappa=0$, which is just the condition for $G:\Sigma\to\mathbb{S}^3$ being a Willmore surface. On the other hand the conformal invariants κ, s and the δ-differential of f satisfy Eq. (41) in which $\overline{h(\xi_1-\xi_2)\kappa}$ is real valued. If f has non-zero parallel mean curvature vector then $\delta = h(\xi_1-\xi_2)dz^2$ is holomorphic by Lemma 4.4. This precisely says that $G:\Sigma\to\mathbb{S}^3_1$ is constrained Willmore.

Conversely if the null Gauss map $G:\Sigma\to\mathbb{S}^3_1$ of f is a constrained Willmore surface then $\kappa_{\bar z\bar z}+\frac{\bar s}{2}\kappa = Re\,(\overline{\eta}\,\kappa)$, for some holomorphic quadratic differential ηdz^2. But κ, s uniquely determine the δ-differential of f by Theorem 4.1, so that $\delta=\eta dz^2$. Therefore δ is holomorphic which implies that f has parallel mean curvature vector by Lemma 4.4. □

5 One-parameter deformations and associated families

5.1 The $\mathbb{S}^1$-deformation of marginally trapped surfaces with non-zero parallel mean curvature

Here we consider the harmonic map equation and its loop group formulation for maps of Riemann surfaces with values in the Grassmann manifold $G_2(\mathbb{R}^5)$ of oriented spacelike 2-planes through the origin of $\mathbb{R}^5_1$. We describe the spectral symmetry of the

Gauss map of marginally trapped surfaces in $\mathbb{S}^4_1$ with parallel mean curvature vector and construct non-trivial one-parameter deformations. A consequence is that marginally trapped surfaces in $\mathbb{S}^4_1$ with non-zero parallel mean curvature always come in one-parameter deformation families called associated families. We use these associated families to obtain their conformal invariants.

We start recalling that $\Lambda^2\mathbb{R}^5_1 = span\{e_i \wedge e_j : 0 \le i < j \le 4\} \cong \mathbb{R}^{10}$, where $\{e_0, \dots e_4\}$ is the canonical orthonormal basis of $\mathbb{R}^5_1$. There is a familiar identification of $\mathfrak{so}(4,1)$ with $\Lambda^2\mathbb{R}^5_1$ via:

$$(x \wedge y)(u) = \langle x, u\rangle y - \langle y, u\rangle x. \tag{43}$$

Under this identification one obtains

$$e_i \wedge e_j = \begin{cases} E_{ji} - E_{ij}, & 0 \le i < j \le 3, \\ E_{i4} + E_{4i}, & 0 \le i < 4, \end{cases} \tag{44}$$

where E_{ij} is the 5×5 matrix with a $+1$ in entry (i, j).

We shall use that $\Lambda^2\mathbb{R}^5_1$ is equipped with the nondegenerate pseudo-riemannian inner product given by:

$$\langle x \wedge y, x' \wedge y'\rangle = \langle x, x'\rangle\langle y, y'\rangle - \langle x, y'\rangle\langle y, x'\rangle. \tag{45}$$

Thus $\Lambda^2(\mathbb{R}^5_1)$ has signature $(6, 4)$ with respect to this inner product since the set $\{e_i \wedge e_j : 0 \le i < j \le 4\}$ is an orthonormal basis of $\Lambda^2(\mathbb{R}^5_1)$, satisfying

$$\|e_i \wedge e_j\|^2 = 1, \quad 0 \le i < j \le 3,$$
$$\|e_i \wedge e_4\|^2 = -1, \quad 0 \le i < 4.$$

A straightforward calculation using (43), (44) and (45) shows that the adjoint representation of $SO(4, 1)$ on bivectors takes the form

$$Ad(F)(x \wedge y) = Fx \wedge Fy, \quad F \in SO(4, 1). \tag{46}$$

It follows from (46) that the inner product (45) is invariant under the adjoint action of $SO(4, 1)$.

Using that $d(Ad) = ad$, (43) and (46) we obtain the expression of the Lie bracket of bivectors:

$$[x \wedge y, x' \wedge y'] = \langle x, x'\rangle\, y \wedge y' - \langle y, x'\rangle\, x \wedge y' + \langle x, y'\rangle\, x' \wedge y - \langle y, y'\rangle\, x' \wedge x. \tag{47}$$

Any oriented spacelike 2-plane P through the origin of $\mathbb{R}^5_1$ can be identified with the bi-vector $x \wedge y \in \Lambda^2\mathbb{R}^5_1$, where $\{x, y\}$ is an oriented orthonormal basis of P. Hence the Grassmann manifold of all oriented spacelike 2-planes of $\mathbb{R}^5_1$ through the origin is given by:

$$G_2(\mathbb{R}^5_1) = \{x \wedge y \in \Lambda^2(\mathbb{R}^5_1) : \langle x, x\rangle = \langle y, y\rangle = 1, \langle x, y\rangle = 0\}.$$

This allows us to view $G_2(\mathbb{R}^5_1)$ as a submanifold of $\mathfrak{so}(4,1) \equiv \Lambda^2(\mathbb{R}^5_1)$ in a natural way. The ambient pseudo-metric on $\mathfrak{so}(4,1) \equiv \Lambda^2(\mathbb{R}^5_1)$ induces a pseudo-metric on $G_2(\mathbb{R}^5_1)$ of signature $(4,2)$ with respect to which $SO(4,1)$ acts transitively by isometries via (46). The inclusion map $G_2(\mathbb{R}^5_1) \hookrightarrow \Lambda^2(\mathbb{R}^5_1)$ is the *standard imbedding* of the pseudo Kaehler symmetric space $G_2(\mathbb{R}^5_1)$.

We fix the base point $E := e_1 \wedge e_2 \in G_2(\mathbb{R}^5_1)$, which identifies via (43) with the matrix $E_{21} - E_{12} \in \mathfrak{so}(4,1)$. Thus $G_2(\mathbb{R}^5_1)$ is diffeomorphic to the quotient $SO_+(4,1)/H$, where H is the connected component of the identity of the isotropy subgroup of E. Let τ be the automorphism of $SO(4,1)$ defined by $\tau(F) = LFL$, where $L := diag(1,-1,-1,1,1) \in SO(4,1)$. Then $Fix(\tau)_0 \subseteq H \subseteq Fix(\tau)$, where $Fix(\tau)_0$ is the connected component of the subgroup of fixed points of τ. The Lie algebra $\mathfrak{so}(4,1)$ then splits into the direct sum of the (± 1)-eigenspaces of $d\tau_e$ which are given respectively by

$$\mathfrak{h} := \left\{ \begin{pmatrix} 0 & 0 & 0 & m & n \\ 0 & 0 & s & 0 & 0 \\ 0 & -s & 0 & 0 & 0 \\ -m & 0 & 0 & 0 & t \\ n & 0 & 0 & t & 0 \end{pmatrix} : s,t,m,n \in \mathbb{R} \right\},$$
$$\mathfrak{m} := \left\{ \begin{pmatrix} 0 & a & b & 0 & 0 \\ -a & 0 & 0 & c & d \\ -b & 0 & 0 & e & k \\ 0 & -c & -e & 0 & 0 \\ 0 & d & k & 0 & 0 \end{pmatrix} : a,b,c,d,e,k \in \mathbb{R} \right\}. \qquad (48)$$

Since $d\tau_e$ is a Lie algebra automorphism it follows that the decomposition $\mathfrak{so}(4,1) = \mathfrak{h} \oplus \mathfrak{m}$ is symmetric, hence reductive since $\mathfrak{h}, \mathfrak{m}$ satisfy the relations:

$$[\mathfrak{h},\mathfrak{m}] \subseteq \mathfrak{m}, \quad [\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{h}, \quad [\mathfrak{m},\mathfrak{m}] \subseteq \mathfrak{h}. \qquad (49)$$

It is not difficult to check that $\mathfrak{h}$ is isomorphic to $\mathfrak{so}(2) \oplus \mathfrak{so}(3,1)$. Also from (48) and (49) it follows that $X \in \mathfrak{h} = Lie(H)$ if and only if $[X, E] = 0$.

Let $\phi : \Sigma \to G_2(\mathbb{R}^5_1)$ be a smooth map and $F : \Sigma \to SO(4,1)$ a locally defined map such that $\phi = Ad(F)E = Fe_1 \wedge Fe_2$. Such F is called a (local) frame of ϕ (if Σ is simply connected then there is always a global frame F of ϕ). From $d(Ad) = ad$ one easily derives the useful identity

$$dAd(F) = Ad(F) \circ ad\,\alpha,$$

in which $\alpha := F^{-1}dF$ is the pull-back of the left Maurer–Cartan one form of $SO(4,1)$ by the frame F. Using this identity we obtain the derivative of ϕ:

$$d\phi = Ad(F)[\alpha, E] = [Ad(F) \circ ad\alpha, Ad(F)E]. \qquad (50)$$

On the other hand we can express $\alpha = Adz + Bd\bar{z}$, where the matrices $A, B \in so(4,1,\mathbb{C})$ are defined by $A := F^{-1}F_z$ and $B := F^{-1}F_{\bar{z}}$. Then the integrabilty condition $F_{z\bar{z}} = F_{\bar{z}z}$ in terms of A, B reads $A_{\bar{z}} - B_z = [A, B]$, which is equivalent to the Maurer–Cartan equation: $d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0$. Recall that the wedge product of two

Lie algebra-valued one forms a, b on Σ is defined by $[a \wedge b](X, Y) := [a(X), b(Y)] - [a(Y), b(X)], \quad X, Y \in T\Sigma$.

Decomposing $A = A_{\mathfrak{h}} + A_{\mathfrak{m}}$ and $B = B_{\mathfrak{h}} + B_{\mathfrak{m}}$, with $A_{\mathfrak{h}}, B_{\mathfrak{h}} \in \mathfrak{h}^{\mathbb{C}}$ and $A_{\mathfrak{m}}, B_{\mathfrak{m}} \in \mathfrak{m}^{\mathbb{C}}$, α can be written as

$$\alpha = \alpha_{\mathfrak{h}} + \alpha_{\mathfrak{m}}, \quad \alpha_{\mathfrak{h}} = A_{\mathfrak{h}} dz + B_{\mathfrak{h}} d\bar{z}, \quad \alpha_{\mathfrak{m}} = A_{\mathfrak{m}} dz + B_{\mathfrak{m}} d\bar{z}, \tag{51}$$

where $\alpha_{\mathfrak{h}} \in \Gamma(\mathfrak{h} \otimes T^*\Sigma)$, and $\alpha_{\mathfrak{m}} \in \Gamma(\mathfrak{m} \otimes T^*\Sigma)$.

Using (50), the above decomposition and $[A_{\mathfrak{h}}, E] = 0$ we obtain

$$\begin{aligned} d\phi(\partial_z) &= Ad(F)[\alpha(\partial_z), E] = Ad(F)[A_{\mathfrak{m}} + A_{\mathfrak{h}}, E] = Ad(F)[A_{\mathfrak{m}}, E] \\ &= [Ad(F)A_{\mathfrak{m}}, \phi]. \end{aligned}$$

By viewing ϕ as a map into the ambient space $\Lambda^2(\mathbb{R}^5_1)$ rather than into $G_2(\mathbb{R}^5_1)$ we may compute the tension of ϕ. Choose any conformal metric g on Σ, then the Levi-Civita connection ∇^g satisfies $\nabla^g_{\partial_{\bar{z}}} \partial_z = 0$. Thus since $[[B_{\mathfrak{m}}, A_{\mathfrak{m}}], E] = 0$, we obtain

$$\begin{aligned} &\partial_{\bar{z}} d\phi(\partial_z) - d\phi(\nabla^g_{\partial_{\bar{z}}} \partial_z) = \partial_{\bar{z}} Ad(F)[A_{\mathfrak{m}}, E] = \\ &Ad(F) \circ ad\, \alpha(\partial_{\bar{z}})([A_{\mathfrak{m}}, E]) + Ad(F)[\partial_{\bar{z}} A_{\mathfrak{m}}, E] = \\ &Ad(F)[B_{\mathfrak{h}} + B_{\mathfrak{m}}, [A_{\mathfrak{m}}, E]] + Ad(F)[\partial_{\bar{z}} A_{\mathfrak{m}}, E] = \\ &Ad(F)[\partial_{\bar{z}} A_{\mathfrak{m}} + [B_{\mathfrak{h}}, A_{\mathfrak{m}}], E] + Ad(F)[B_{\mathfrak{m}}, [A_{\mathfrak{m}}, E]]. \end{aligned}$$

Taking into account that

$$\begin{aligned} &Ad(F)[\partial_{\bar{z}} A_{\mathfrak{m}} + [B_{\mathfrak{h}}, A_{\mathfrak{m}}], E] \in T_\phi G_2(\mathbb{R}^5_1)^{\mathbb{C}}, \\ &Ad(F)[B_{\mathfrak{m}}, [A_{\mathfrak{m}}, E]] \in T^\perp_\phi G_2(\mathbb{R}^5_1)^{\mathbb{C}}, \end{aligned}$$

then up to a non-zero multiple depending on the conformal metric g the tension of ϕ is given by

$$(\partial_{\bar{z}} d\phi(\partial_z))^T = Ad(F)[\ \partial_{\bar{z}} A_{\mathfrak{m}} + [B_{\mathfrak{h}}, A_{\mathfrak{m}}], E], \tag{52}$$

where $(,)^T$ denotes the component in $T_\phi G_2(\mathbb{R}^5_1)$. It follows that $\phi : \Sigma \to G_2(\mathbb{R}^5_1)$ is harmonic if and only if

$$\partial_{\bar{z}} A_{\mathfrak{m}} + [B_{\mathfrak{h}}, A_{\mathfrak{m}}] = 0. \tag{53}$$

According to the decomposition $T^{\mathbb{C}}\Sigma = T'\Sigma \oplus T''\Sigma$, $\alpha_{\mathfrak{m}}$ and $\alpha_{\mathfrak{h}}$ decompose into its $(1, 0)$ and $(0, 1)$ parts respectively: $\alpha_{\mathfrak{m}} = \alpha'_{\mathfrak{m}} + \alpha''_{\mathfrak{m}}$ and $\alpha_{\mathfrak{h}} = \alpha'_{\mathfrak{h}} + \alpha''_{\mathfrak{h}}$, where $\overline{\alpha'_{\mathfrak{m}}} = \alpha''_{\mathfrak{m}}$ and $\overline{\alpha'_{\mathfrak{h}}} = \alpha''_{\mathfrak{h}}$. From (51) it follows that

$$\alpha'_{\mathfrak{m}} = A_{\mathfrak{m}} dz, \quad \alpha''_{\mathfrak{m}} = B_{\mathfrak{m}} d\bar{z}, \quad \alpha_{\mathfrak{h}} = A_{\mathfrak{h}} dz + B_{\mathfrak{h}} d\bar{z}. \tag{54}$$

Therefore $d\alpha'_{\mathfrak{m}} = \bar{\partial}\alpha'_{\mathfrak{m}} = -\partial_{\bar{z}} A_{\mathfrak{m}} dz \wedge d\bar{z}$, where $d = \partial + \bar{\partial}$. Also

$$[\alpha_{\mathfrak{h}} \wedge \alpha'_{\mathfrak{m}}] = [(A_{\mathfrak{h}} dz + B_{\mathfrak{h}} d\bar{z}) \wedge A_{\mathfrak{m}} dz] = -[B_{\mathfrak{h}}, A_{\mathfrak{m}}] dz \wedge d\bar{z}.$$

Thus the harmonic map equation (53) becomes

$$d\alpha'_{\mathfrak{m}} + [\alpha_{\mathfrak{h}} \wedge \alpha'_{\mathfrak{m}}] = 0. \tag{55}$$

According to the decomposition $\alpha = \alpha_{\mathfrak{h}} + \alpha_{\mathfrak{m}}$, the Maurer–Cartan equation $d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0$, splits up into $\mathfrak{m}$ and $\mathfrak{h}$ parts:

$$\begin{aligned} &d\alpha'_{\mathfrak{m}} + [\alpha_{\mathfrak{h}} \wedge \alpha'_{\mathfrak{m}}] + d\alpha''_{\mathfrak{m}} + [\alpha_{\mathfrak{h}} \wedge \alpha''_{\mathfrak{m}}] = 0, \\ &d\alpha_{\mathfrak{h}} + \tfrac{1}{2}[\alpha_{\mathfrak{h}} \wedge \alpha_{\mathfrak{h}}] + [\alpha'_{\mathfrak{m}} \wedge \alpha''_{\mathfrak{m}}] = 0. \end{aligned} \tag{56}$$

Combining the above equations with (55) one obtains

$$\begin{aligned} &d\alpha''_{\mathfrak{m}} + [\alpha_{\mathfrak{h}} \wedge \alpha''_{\mathfrak{m}}] = 0, \\ &d\alpha_{\mathfrak{h}} + \tfrac{1}{2}[\alpha_{\mathfrak{h}} \wedge \alpha_{\mathfrak{h}}] + [\alpha'_{\mathfrak{m}} \wedge \alpha''_{\mathfrak{m}}] = 0. \end{aligned} \tag{57}$$

Let $\lambda \in \mathbb{C}$, with $|\lambda| = 1$ and consider the one-parameter family of one forms defined by

$$\alpha_\lambda := \lambda^{-1}\alpha'_{\mathfrak{m}} + \alpha_{\mathfrak{h}} + \lambda\alpha''_{\mathfrak{m}}. \tag{58}$$

Note that since $\overline{\alpha'_{\mathfrak{m}}} = \alpha''_{\mathfrak{m}}$ and $\overline{\alpha_{\mathfrak{h}}} = \alpha_{\mathfrak{h}}$, then α_λ is $\mathfrak{g}$-valued for every $\lambda \in \mathbb{S}^1$. Comparing coefficients of λ, it follows that equations (57) above hold for α if and only if the one-parameter family α_λ satisfies

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0, \quad \forall \lambda \in \mathbb{S}^1. \tag{59}$$

It follows that $\phi : \Sigma \to G_2(\mathbb{R}^5_1)$ is harmonic if and only if (59) holds. Equivalently ϕ is harmonic if and only if $d + \alpha_\lambda$, $\lambda \in \mathbb{S}^1$ is a loop of flat connections on the trivial bundle $\Sigma \times \mathfrak{so}(4, 1)$.

Now let $f : \Sigma \to \mathbb{S}^4_1$ be an conformally immersed non-isotropic marginally trapped surface satisfying $\langle f_z, f_{\bar{z}} \rangle = e^{2u}$ where u is a local conformal parameter. Then the Gauss map of f is given with respect to a local coordinate $z = x + iy$ by

$$\gamma_f = \frac{e^{-2u}}{2} f_x \wedge f_y = -ie^{-2u} f_z \wedge f_{\bar{z}}. \tag{60}$$

A frame $F = (F_0, F_1, F_2, N_1, N_2) \in SO(4, 1)$ (in ordered columns notation) is *adapted* to the surface f or f-adapted if $F_0 = f$ and F_1, F_2 span the tangent space of the immersed surface. For such frame F, the fields N_1, N_2 are sections of $\nu(f)$ i.e. they form a lorentzian orthonormal frame of normal vectors along f. Since f is conformal and F is f-adapted we can rotate within the tangent plane $df(T\Sigma) = span\{F_1, F_2\}$ if necessary, so that $f_z = \frac{e^u}{\sqrt{2}}(F_1 - iF_2)$. In this case $F_1 = \frac{e^{-u}}{\sqrt{2}} f_x$, $F_2 = \frac{e^{-u}}{\sqrt{2}} f_y$. Then $\gamma_f = F_1 \wedge F_2$, so that F also frames the Gauss map γ_f.

We assume that the f-adapted frame $F \in SO_+(4, 1)$ is defined on the universal convering $\tilde{\Sigma}$. In terms of F the structure equations (3) of the marginally trapped immersed surface f with respect to a coordinate z can be written as $F_z = F.A$, where the matrix A is given by

$$A = \begin{pmatrix} 0 & -\frac{e^u}{\sqrt{2}} & i\frac{e^u}{\sqrt{2}} & 0 & 0 \\ \frac{e^u}{\sqrt{2}} & 0 & iu_z & -a_1 & a_2 \\ -i\frac{e^u}{\sqrt{2}} & -iu_z & 0 & -ib_1 & ib_2 \\ 0 & a_1 & ib_1 & 0 & \sigma \\ 0 & a_2 & ib_2 & \sigma & 0 \end{pmatrix}, \tag{61}$$

with coefficients

$$\begin{aligned} a_1 &= \tfrac{e^u h + e^{-u}\xi_1}{\sqrt{2}}, \quad b_1 = \tfrac{-e^u h + e^{-u}\xi_1}{\sqrt{2}}, \\ a_2 &= \tfrac{e^u h + e^{-u}\xi_2}{\sqrt{2}}, \quad b_2 = \tfrac{-e^u h + e^{-u}\xi_2}{\sqrt{2}}. \end{aligned} \tag{62}$$

In this case the equations of Gauss, Codazzi and Ricci (4) encoded in the Maurer–Cartan equation $d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0$, are just the integrability conditions for the existence of an f-adapted frame F solving $F^{-1}dF = \alpha$.

Decompose $A = A_{\mathfrak{m}} + A_{\mathfrak{h}}$, and $B = B_{\mathfrak{m}} + B_{\mathfrak{h}}$, where

$$A_{\mathfrak{m}} = \begin{pmatrix} 0 & -\frac{e^u}{\sqrt{2}} & i\frac{e^u}{\sqrt{2}} & 0 & 0 \\ \frac{e^u}{\sqrt{2}} & 0 & 0 & -a_1 & a_2 \\ -i\frac{e^u}{\sqrt{2}} & 0 & 0 & -ib_1 & ib_2 \\ 0 & a_1 & ib_1 & 0 & 0 \\ 0 & a_2 & ib_2 & 0 & 0 \end{pmatrix}, \quad B_{\mathfrak{m}} = \overline{A}_{\mathfrak{m}}, \tag{63}$$

$$A_{\mathfrak{h}} = diag(0, \left(\begin{smallmatrix} 0 & iu_z \\ -iu_z & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & \sigma \\ \sigma & 0 \end{smallmatrix}\right)), \quad B_{\mathfrak{h}} = \overline{A}_{\mathfrak{h}}. \tag{64}$$

By (52) the tension of the Gauss map γ_f, up to a non-zero multiple depending on the f-induced metric, identifies with the $\mathfrak{m}^{\mathbb{C}}$-valued matrix

$$\partial_{\bar{z}} A_{\mathfrak{m}} + [B_{\mathfrak{h}}, A_{\mathfrak{m}}] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathtt{A}_1 & \mathtt{A}_2 \\ 0 & 0 & 0 & -i\mathtt{B}_1 & i\mathtt{B}_2 \\ 0 & \mathtt{A}_1 & i\mathtt{B}_1 & 0 & 0 \\ 0 & \mathtt{A}_2 & i\mathtt{B}_2 & 0 & 0 \end{pmatrix},$$

where the coefficients $\mathtt{A}_1, \mathtt{A}_2, \mathtt{B}_1, \mathtt{B}_2$ are given by

$$\begin{aligned} \mathtt{A}_1 &= a_{1\bar{z}} + \bar{\sigma} a_2 + u_{\bar{z}} b_1, \quad \mathtt{A}_2 = a_{2\bar{z}} + \bar{\sigma} a_1 + u_{\bar{z}} b_2, \\ \mathtt{B}_1 &= b_{1\bar{z}} + \bar{\sigma} b_2 + u_{\bar{z}} a_1, \quad \mathtt{B}_2 = b_{2\bar{z}} + \bar{\sigma} b_1 + u_{\bar{z}} a_2. \end{aligned}$$

By a straightforward calculation using (62) and Codazzi's equations (4) we obtain

$$\mathtt{A}_1 = \mathtt{A}_2 = \frac{e^u}{\sqrt{2}} 2Re\{h_z + \sigma h\},$$
$$\mathtt{B}_1 = \mathtt{B}_2 = \frac{e^u}{\sqrt{2}} 2i\,Im\{h_z + \sigma h\},$$

Thus the Gauss map γ_f is a harmonic map if and only if $h_z + \sigma h = 0$, which by (11) is just the condition for f having parallel mean curvature vector field. We have thus proved the following particular case of the characterization due to Ruh–Vilms of submanifods with parallel mean curvature vector in riemannian space forms (Ruh and Vilms 1970):

Lemma 5.1 *Let $f : \Sigma \to \mathbb{S}^4_1$ be a conformally immersed marginally trapped surface. Then the Gauss map $\gamma_f : \Sigma \to G_2(\mathbb{R}^5_1)$ of f is harmonic if and only if f has parallel mean curvature.*

In what follows we assume that the conformally immersed marginally trapped f has non-zero parallel mean curvature vector. Then the normal bundle $\nu(f)$ is flat (Rahim Elghanmi 1996) and we can assume that the normal (sub) frame $\{N_1, N_2\}$ of the f-adapted frame $F : \Sigma \to SO_+(4, 1)$ is positively oriented and $\nabla^\perp$-parallel along f.

Fixing a point $x_0 \in \tilde{\Sigma}$ we integrate for each $\lambda \in \mathbb{S}^1$ the differential equation

$$dF^\lambda = F^\lambda \alpha_\lambda, \tag{65}$$

with initial condition $F^\lambda(x_0) = F(x_0) \in H$. We obtain a solution $F^\lambda : \tilde{\Sigma} \to SO_+(4, 1)$, (hence a local solution around any point of Σ) which is called an *extended frame*. According to Burstall and Pedit (1995) it is possible to choose the constants of integration so that F^λ depends smoothly on $\lambda \in \mathbb{S}^1$. In column notation,

$$F^\lambda = (F_0^\lambda, F_1^\lambda, F_2^\lambda, N_1^\lambda, N_2^\lambda).$$

Since $\alpha_{\{\lambda=1\}} = \alpha$, the extended frame satisfies $F^{\{\lambda=1\}}(x) = F(x), \forall x \in \tilde{\Sigma}$. In particular $N_1^{\{\lambda=1\}} = N_1$, and $N_2^{\{\lambda=1\}} = N_2$, where $\{N_1, N_2\} \subset \Gamma(\nu(f))$ is positively oriented by assumption. Thus an elementary argument shows that $\{N_1^\lambda, N_2^\lambda\}$ is positively oriented $\forall \lambda \in \mathbb{S}^1$.

We define $f^\lambda := F_0^\lambda = F^\lambda e_0$, i.e. the first column of the extended frame F^λ. Then f^λ is a one parameter deformation of f since at $\lambda = 1$ we recover f: $F^{\lambda=1} e_0 = F.e_0 = F_0 = f$. We call $\{f^\lambda, \lambda \in \mathbb{S}^1\}$ the *associated family* of the marginally trapped surface f. From (58) and (65) we get

$$f_z^\lambda := F_z^\lambda e_0 = F^\lambda(\lambda^{-1} A_{\mathfrak{m}} + A_{\mathfrak{h}}) e_0 = \lambda^{-1} \frac{e^u}{\sqrt{2}} F^\lambda(e_1 - ie_2) = \lambda^{-1} \frac{e^u}{\sqrt{2}} (F_1^\lambda - iF_2^\lambda), \tag{66}$$

hence F^λ is adapted to f^λ. From (66) we obtain:

$$\langle f_z^\lambda, f_z^\lambda\rangle = \left\langle \lambda^{-1}\frac{e^u}{\sqrt{2}}(e_1 - ie_2), \lambda^{-1}\frac{e^u}{\sqrt{2}}(e_1 - ie_2)\right\rangle = 0.$$
$$\langle f_z^\lambda, f_{\bar z}^\lambda\rangle = \langle F^\lambda(\lambda^{-1}A_{\mathfrak{m}} + A_{\mathfrak{h}})e_0, F^\lambda(\lambda B_{\mathfrak{m}} + B_{\mathfrak{h}})e_0\rangle$$
$$= \left\langle \lambda^{-1}\frac{e^u}{\sqrt{2}}(e_1 - ie_2), \lambda\frac{e^u}{\sqrt{2}}(e_1 + ie_2)\right\rangle = e^{2u}. \quad (67)$$

Thus for each $\lambda \in \mathbb{S}^1$, f^λ is a conformally immersed surface (hence spacelike) inducing the same conformal metric.

Consider the one-parameter family of maps goven by $Ad(F^\lambda)E = F_1^\lambda \wedge F_2^\lambda$. Since the extended frame satisfies $F^{\{\lambda=1\}}(x) = F(x), \forall x \in \tilde\Sigma$, it follows that $F_1^{\{\lambda=1\}} \wedge F_2^{\{\lambda=1\}} = F_1 \wedge F_2 = \gamma_f$. This shows that $Ad(F^\lambda)E$ is a one parameter deformation of γ_f. Hence we define $(\gamma_f)^\lambda := Ad(F^\lambda)E$, which is the associated family of the harmonic map γ_f.

Note that by (60) the Gauss map of f^λ is given by $\gamma_{f^\lambda} = -ie^{-2u} f_z^\lambda \wedge f_{\bar z}^\lambda$. Thus from (66) it follows that $(\gamma_f)^\lambda = \gamma_{f^\lambda}$.

Due to $(\alpha_\lambda)'_{\mathfrak{m}} = \lambda^{-1}\alpha'_{\mathfrak{m}}$, $(\alpha_\lambda)''_{\mathfrak{m}} = \lambda\alpha''_{\mathfrak{m}}$, and $(\alpha_\lambda)_{\mathfrak{h}} = \alpha_{\mathfrak{h}}$, the one form α_λ satisfies Eq. (55) for any $\lambda \in \mathbb{S}^1$. Thus each $(\gamma_f)^\lambda : \Sigma \to G_2(\mathbb{R}_1^5)$ is harmonic and so each f^λ has parallel mean curvature vector.

We claim that f^λ is a non-isotropic marginally trapped surface with non-zero mean curvature vector for any $\lambda \in \mathbb{S}^1$. Denote by $\vec{H}_\lambda$ the mean curvature vector of f^λ. Since f^λ is conformal and spacelike, it follows that

$$f_{z\bar z}^\lambda = -e^{2u} f^\lambda + e^{2u}\vec{H}_\lambda, \quad (68)$$

hence from (6) we obtain

$$\vec{H}_\lambda = e^{-2u}\langle f_{z\bar z}^\lambda, N_1^\lambda\rangle N_1^\lambda - e^{-2u}\langle f_{z\bar z}^\lambda, N_2^\lambda\rangle N_2^\lambda. \quad (69)$$

On the other hand from (65) the structure equations of f^λ are given by $F_z^\lambda = F^\lambda(\lambda^{-1}A_{\mathfrak{m}} + A_{\mathfrak{h}})$, which is equivalent to the system

$$f_z^\lambda = \frac{1}{\lambda}\frac{e^u}{\sqrt{2}}F_1^\lambda - i\frac{1}{\lambda}\frac{e^u}{\sqrt{2}}F_2^\lambda,$$
$$\partial_z F_1^\lambda = -\frac{1}{\lambda}\frac{e^u}{\sqrt{2}}f^\lambda - iu_z F_2^\lambda + \frac{1}{\lambda}a_1 N_1^\lambda + \frac{1}{\lambda}a_2 N_2^\lambda,$$
$$\partial_z F_2^\lambda = i\frac{1}{\lambda}\frac{e^u}{\sqrt{2}}f^\lambda + iu_z F_1^\lambda + i\frac{1}{\lambda}b_1 N_1^\lambda + i\frac{1}{\lambda}b_2 N_2^\lambda,$$
$$\partial_z N_1^\lambda = -\frac{1}{\lambda}a_1 F_1^\lambda - i\frac{1}{\lambda}b_1 F_2^\lambda + \sigma N_1^\lambda,$$
$$\partial_z N_2^\lambda = \frac{1}{\lambda}a_2 F_1^\lambda + i\frac{1}{\lambda}b_2 F_2^\lambda + \sigma N_2^\lambda, \quad (70)$$

from which we obtain

$$\langle f^{\lambda}_{z\bar{z}}, N^{\lambda}_1\rangle = -\langle f^{\lambda}_{\bar{z}}, \partial_z N^{\lambda}_1\rangle = (a_1 - b_1)\frac{e^u}{\sqrt{2}} = e^{2u}h,$$
$$\langle f^{\lambda}_{z\bar{z}}, N^{\lambda}_2\rangle = -\langle f^{\lambda}_{\bar{z}}, \partial_z N^{\lambda}_2\rangle = (b_2 - a_2)\frac{e^u}{\sqrt{2}} = -e^{2u}h.$$

Thus from equation (69) we obtain $\vec{H}_\lambda = h(N^{\lambda}_1 + N^{\lambda}_2)$, hence $h^\lambda = h \neq 0$, which shows that f^λ is marginally trapped with non-zero mean curvature vector for every $\lambda \in \mathbb{S}^1$.

On the other hand since $\xi^{\lambda}_1 = \langle f^{\lambda}_{zz}, N^{\lambda}_1\rangle$ and $\xi^{\lambda}_2 = -\langle f^{\lambda}_{zz}, N^{\lambda}_2\rangle$, then from (70) we obtain

$$\xi^{\lambda}_1 = \langle f^{\lambda}_{zz}, N^{\lambda}_1\rangle = -\langle f^{\lambda}_{z}, \partial_z N^{\lambda}_1\rangle = \lambda^{-2}\frac{e^u}{\sqrt{2}}(a_1 + b_1) = \lambda^{-2}\xi_1,$$
$$\xi^{\lambda}_2 = -\langle f^{\lambda}_{zz}, N^{\lambda}_2\rangle = \langle f^{\lambda}_{z}, \partial_z N^{\lambda}_2\rangle = \lambda^{-2}\frac{e^u}{\sqrt{2}}(a_2 + b_2) = \lambda^{-2}\xi_2. \tag{71}$$

Therefore the (2, 0) part of the second fundamental form of f_λ is given by,

$$I\!I^{\lambda}(\partial_z, \partial_z) = \lambda^{-2}\xi_1 N^{\lambda}_1 + \lambda^{-2}\xi_2 N^{\lambda}_2. \tag{72}$$

From the above expression results that $Q_\lambda = \lambda^{-2}Q$, where $Q_\lambda = \langle f^{\lambda}_{zz}, f^{\lambda}_{zz}\rangle dz^4$. Hence f^λ is non-isotropic for every $\lambda \in \mathbb{S}^1$. We collect the above facts in the following,

Proposition 5.1 *Let $f : \Sigma \to \mathbb{S}^4_1$ be a non-isotropic conformal marginally trapped immersion with non-zero parallel mean curvature vector and Gauss map γ_f. Let f^λ its associated family obtained by integration of* (65)*, which is defined on a simply connected open neighbourhood of each point of Σ. Then each member f^λ is a non-isotropic conformal immersion inducing the same conformal metric for any $\lambda \in \mathbb{S}^1$ with non-zero parallel mean curvature vector. Moreover, the Gauss map of f^λ is the λ-deformation of the Gauss map of f, i.e. $\gamma_{f^\lambda} = (\gamma_f)^\lambda$.*

Next we describe the conformal invariants κ_λ, s_λ and δ-differential of the associated family f^λ obtained above. First note that the Hopf quadratic differential $q = (\xi_1 - \xi_2)dz^2$ of f is never zero on Σ and is holomorphic by Lemma 4.2 since f has flat normal bundle. Thus for any point $x \in \Sigma$ there is a local coordinate z such that $q = c dz^2$, for a non-zero real constant c. By (31) κ is real in the same coordinate z and so the null Gauss map $G : \Sigma \to \mathbb{S}^3$ of f is isothermic.

From (71) and (72) we obtain the Hopf differential of f^λ:

$$q_\lambda = \lambda^{-2} c dz^2 = \lambda^{-2} q. \tag{73}$$

Since $h_\lambda = h$, from the above expression we obtain

$$\delta_\lambda = h q_\lambda = \lambda^{-2} c h dz^2 = \lambda^{-2}\delta. \tag{74}$$

In polar form the Hopf differential q_λ is given by, $q_\lambda = |c|e^{i\theta(\lambda)}dz^2 = \lambda^{-2}|c|e^{i\theta}dz^2$. Thus $e^{i\theta(\lambda)} = \lambda^{-2}e^{i\theta}$ so that if $\lambda = e^{i\varphi}$ then

$$\theta(\lambda) = \theta - 2\varphi. \tag{75}$$

Note that λ does not depend on z, an consequently $\varphi_z = 0$. Hence $\theta(\lambda)_z = \theta_z$, and $\theta(\lambda)_{zz} = \theta_{zz}$. Taking this into account then formula (39) in terms of the conformal invariants κ_λ, s_λ and δ_λ becomes:

$$ch\lambda^{-2} = ((u - i\theta)_z)^2 + (u - i\theta)_{zz} + \frac{s_\lambda}{2}. \tag{76}$$

Combining the above equation with (39) gives the schwartzian derivative s_λ of G_λ:

$$s_\lambda = s + 2(\lambda^{-2} - 1)ch. \tag{77}$$

Also from (32) κ_λ identifies with $\frac{e^{u+i\theta(\lambda)}}{\sqrt{2}}$, thus from (75) we obtain

$$\kappa_\lambda = \frac{e^{u+i(\theta-2\varphi)}}{\sqrt{2}} = \lambda^{-2}\kappa. \tag{78}$$

A straightforward computation using (76), (77) and (78) shows that κ_λ, s_λ, δ_λ obey the fundamental equation (41) namely,

$$(\kappa_\lambda)_{\bar z\bar z} + \frac{\overline{s_\lambda}}{2}\kappa_\lambda = ch\overline{\lambda^{-2}}\kappa_\lambda, \quad \forall\lambda \in \mathbb{S}^1,$$

in which $ch\overline{\lambda^{-2}}\kappa_\lambda = ch\kappa$, hence it is real valued for every $\lambda \in \mathbb{S}^1$. Consequently κ_λ, s_λ obey the conformal Codazzi equation:

$$Im\left((\kappa_\lambda)_{\bar z\bar z} + \frac{\overline{s_\lambda}}{2}\kappa_\lambda\right) = 0, \quad \forall\lambda \in \mathbb{S}^1.$$

Since f has parallel mean curvature vector, δ is holomorphic. Thus from (76), (77), (78) it easily follows that κ_λ, s_λ obey the conformal Gauss equation:

$$\frac{(s_\lambda)_{\bar z}}{2} = 3(\overline{\kappa_\lambda})_z.\kappa_\lambda + \overline{\kappa_\lambda}(\kappa_\lambda)_z.$$

Since $\lambda = 0$ does not depend on z and δ is holomorphic, then $\delta_\lambda = \lambda^{-2}\delta$ is holomorphic for any $\lambda \in \mathbb{S}^1$. We have thus proved the following

Proposition 5.2 *Let f^λ be the associated family of a non-isotropic marginally trapped surface $f : \Sigma \to \mathbb{S}^4_1$ with non-zero parallel mean curvature vector. Then for any $\lambda \in \mathbb{S}^1$ the conformal invariants and δ-differential of f^λ are given by*

$$\kappa_\lambda = \lambda^{-2}\kappa, \quad s_\lambda = s + 2(\lambda^{-2} - 1)ch, \quad \delta_\lambda = \lambda^{-2}\delta, \tag{79}$$

where $q = cdz^2$ and $\delta = chdz^2$.

Moreover, the system consisting of (41) *and the conformal Gauss and Codazzi equations* (17) *is invariant under the spectral symmetry determined by* (79).

Note that as consequence of (79) the members of the associated family f^λ are non-congruent, hence the deformation $f \mapsto f^\lambda$ is non-trivial. It also follows that the isothermic condition is preserved by the spectral symmetry (79): if κ is real for some coordinate z then in the new coordinate $w = \frac{1}{\lambda} z\, \kappa_\lambda$ is real since $\kappa_\lambda dz^2 = \kappa dw^2$.

Remark 5.1 In Burstall et al. (2002) the authors obtain the following slightly different spectral symmetry for the conformal Gauss and Codazzi equations of a constrained Willmore surface $\psi : \Sigma \to \mathbb{S}^3$:

$$\kappa_\lambda = \lambda\kappa, \quad s_\lambda = s + (\lambda^2 - 1)\eta, \quad \eta_\lambda = \lambda^2\eta, \tag{80}$$

where ηdz^2 is an holomorphic quadratic differential satisfying $\kappa_{\bar{z}\bar{z}} + \frac{\bar{s}}{2}\kappa = Re(\bar{\eta}\kappa)$.

5.2 The Calapso-Bianchi associated family of marginally trapped surfaces with flat normal bundle

We construct here an integrable deformation of non-isotropic marginally trapped surfaces with flat normal bundle which is related to the so-called Calapso-Bianchi T-transform of isothermic surfaces in $\mathbb{S}^3$ (Burstall et al. 2002). The class of marginally trapped surfaces with flat normal bundle in $\mathbb{S}^4_1$ includes those with non-zero parallel mean curvature vector and also the spacelike isothermic surfaces introduced by Wang (2012).

Recall that a conformally immersed surface $\psi : \Sigma \to \mathbb{S}^3$ is *isothermic* if away from umbilics, it can be conformally parameterized by its curvature lines. In terms of its conformal invariants a surface ψ is isothermic if each point in Σ has a coordinate z for which κ is real: $\kappa = \overline{\kappa}$ (Burstall et al. 2002; Ma 2005). In this case the conformal Gauss and Codazzi's equations (17) away from umbilic points reduce to

$$\begin{aligned} s_{\bar{z}} &= 4(\kappa^2)_z, \\ Im\left(\kappa_{\bar{z}\bar{z}} + \frac{1}{2}\bar{s}\kappa\right) &= 0. \end{aligned} \tag{81}$$

Thus away from umbilic points κ is non-zero and so both equations combine into Calapso's equation: $\Delta(\frac{\kappa_{xy}}{\kappa}) + 8(\kappa^2)_{xy} = 0$. The Calapso-Bianchi T-transform acts on an isothermic surface $\psi : \Sigma \to \mathbb{S}^3$ by deforming the schwartzian s and keeping κ unchanged:

$$s_t = s + t, \quad \kappa_t = \kappa, \quad t \in \mathbb{R}, \tag{82}$$

thus giving rise to the so-called associated family ψ_t (Burstall et al. 2002).

Let $f : \Sigma \to \mathbb{S}^4_1$ be a non-isotropic marginally trapped surface with flat normal bundle. Then by Lemma 4.2 for every point $x \in \Sigma$ there is a local coordinate z such that $q = cdz^2$ for a non-zero real constant c. Thus κ is real in the same coordinate and so the null Gauss map $G : \Sigma \to \mathbb{S}^3$ of f is isothermic. Conversely, if G is isothermic, then κ and so q is real in some coordinate z. Hence f has flat normal bundle if and only if q is constant i.e. $q = cdz^2$ for some non-zero real constant c.

The structure equations of f reads (3) in which $\xi_1 = \xi + c, \xi_2 = \xi, \sigma = 0$ and $h_1 = h_2 = h$, where the positively oriented orthonormal frame $\{N_1, N_2\}$ is $\nabla^\perp$-parallel. The compatibility equations (4) reduce in this case to

$$\begin{aligned} 2u_{\bar{z}z} &= -e^{2u} + e^{-2u}(2c\text{Re}(\xi) + c^2), \\ \xi_{\bar{z}} &= e^{2u}h_z. \\ 0 &= Im((\xi + c)\xi), \end{aligned} \tag{83}$$

where $2\xi c + c^2 \neq 0$ since f is non-isotropic. If h is a non-zero constant, then f has non-zero parallel mean curvature vector field and its null Gauss map $G : \Sigma \to \mathbb{S}^3$ is isothermic and constrained Willmore. On the other hand if h is a non-constant function satisfying (83), then f has flat normal bundle and non-parallel mean curvature vector field.

Since our considerations are local we consider an f-adapted frame $F = (F_0, F_1, F_2, N_1, N_2) \in SO_+(4, 1)$ defined on the universal covering space $\tilde{\Sigma}$. Then the structure equations of f read $F_z = FA$, where the coefficients of the matrix A in (61) are given in this case by

$$\begin{aligned} a_1 &= \tfrac{e^{-u}(\xi+c)+e^u h}{\sqrt{2}}, \ b_1 = \tfrac{e^{-u}(\xi+c)-e^u h}{\sqrt{2}}, \ \sigma = 0, \\ a_2 &= \tfrac{e^{-u}\xi+e^u h}{\sqrt{2}}, \qquad b_2 = \tfrac{e^{-u}\xi-e^u h}{\sqrt{2}}. \end{aligned}$$

We now introduce a one-parameter family of matrices given by

$$A^t = \begin{pmatrix} 0 & -\frac{e^u}{\sqrt{2}} & i\frac{e^u}{\sqrt{2}} & 0 & 0 \\ \frac{e^u}{\sqrt{2}} & 0 & iu_z & -a_1^t & a_2^t \\ -i\frac{e^u}{\sqrt{2}} & -iu_z & 0 & -ib_1^t & ib_2^t \\ 0 & a_1^t & ib_1^t & 0 & 0 \\ 0 & a_2^t & ib_2^t & 0 & 0 \end{pmatrix}, B^t = \overline{A^t} \in \mathfrak{so}(4, 1)^{\mathbb{C}}, \quad t \in \mathbb{R}, \tag{84}$$

with coefficients

$$\begin{aligned} a_1^t &= \tfrac{e^{-u}(\xi+c)+e^u h^t}{\sqrt{2}}, \ b_1^t = \tfrac{e^{-u}(\xi+c)-e^u h^t}{\sqrt{2}}, \\ a_2^t &= \tfrac{e^{-u}\xi+e^u h^t}{\sqrt{2}}, \qquad b_2^t = \tfrac{e^{-u}\xi-e^u h^t}{\sqrt{2}}, \end{aligned} \tag{85}$$

where

$$h^t := h + \frac{t}{2c}, \quad c \in \mathbb{R}^\times, \quad t \in \mathbb{R}. \tag{86}$$

Note that for $t = 0$ we recover A, i.e. $A^{t=0} = A$.

Lemma 5.2 *Define a one parameter family of* $\mathfrak{so}(4, 1)$*-valued one-forms by*

$$\alpha_t := A^t dz + B^t d\bar{z}, \quad t \in \mathbb{R}. \tag{87}$$

Then α_t *coincides with* α *for* $t = 0$ *and it satisfies the Maurer–Cartan equation*

$$d\alpha_t + \frac{1}{2}[\alpha_t \wedge \alpha_t] = 0, \quad \forall t \in \mathbb{R}, \tag{88}$$

if and only if u, ξ, h *satisfy* (83).

Proof Since $B^t = \overline{A^t}$ then α_t is $\mathfrak{so}(4,1)$-valued for every $t \in \mathbb{R}$. On the other hand $d\alpha_t + \frac{1}{2}[\alpha_t \wedge \alpha_t] = 0$ is equivalent to $(A^t)_{\bar{z}} - (B^t)_z = [A^t, B^t]$ which in turn is equivalent to

$$\begin{aligned} 2u_{\bar{z}z} &= -e^{2u} + e^{-2u}(2c\mathrm{Re}(\xi) + c^2), \\ \xi_{\bar{z}} &= e^{2u}(h^t)_z. \\ 0 &= Im((\xi + c)\xi). \end{aligned}$$

Since $(h^t)_z = h_z$ for any $t \in \mathbb{R}$, the above system is invariant under the symmetry (86) and it is equivalent to (83). □

Since we work locally, we may transfer the situation to the universal covering space $\tilde{\Sigma}$ of Σ (note that the case $\tilde{\Sigma} = \mathbb{S}^2$ is excluded, otherwise being q holomorphic it would vanish). Thus we can integrate the Maurer–Cartan equation (88) on $\tilde{\Sigma}$ for each t, obtaining a solution $F^t : \tilde{\Sigma} \to SO_+(4,1)$, which is unique up to left translation by a constant element in $SO_+(4,1)$. Thus F^t satisfies

$$(F^t)^{-1} dF^t = \alpha_t, \quad F^0 = F, \tag{89}$$

since $\alpha_0 = \alpha$. According to Burstall and Pedit (1995) and Ferus and Pedit (1996) it is possible to choose the constants of integration so that $t \mapsto F^t(x)$ is C^∞ for every $x \in \tilde{\Sigma}$. Denote by $F^t := (F_0^t, F_1^t, F_2^t, N_1^t, N_2^t)$ in column notation. Since $N_2^0 = N_2$ is future pointing, then by continuity N_2^t is future pointing for every t. Moreover, since $\{N_1^0, N_2^0\} = \{N_1, N_2\}$ is positively oriented, then an elementary continuity argument shows that $\{N_1^t, N_2^t\}$ is positively oriented for every $t \in \mathbb{R}$.

Define $f^t := F^t.e_0$, the first column of F^t, then

$$f_z^t = F_z^t e_0 = F^t A^t.(e_1 - ie_2) = \frac{e^u}{\sqrt{2}} F^t(e_1 - ie_2), \tag{90}$$

from which we compute

$$\begin{aligned} \langle f_z^t, f_{\bar{z}}^t \rangle &= \frac{e^{2u}}{2} \langle F^t(e_1 - ie_2), F^t(e_1 + ie_2) \rangle = e^{2u}, \\ \langle f_z^t, f_z^t \rangle &= \frac{e^{2u}}{2} \langle F^t(e_1 - ie_2), F^t(e_1 - ie_2) \rangle = 0, \end{aligned}$$

hence f^t is a conformal spacelike immersion which induces the same (conformal) metric for any t. Since $f^{t=0} = f$, f^t is a one parameter deformation of f. Also from (89) and (90) we obtain

$$f^t_{z\bar{z}} = u_{\bar{z}} f^t_z + \frac{e^u}{\sqrt{2}} F^t B(e_1 - ie_2), \quad f^t_{zz} = u_z f_z + \frac{e^u}{\sqrt{2}} F^t A(e_1 - ie_2),$$

which, from the structure of the matrices A^t, B^t, become,

$$f^t_{z\bar{z}} = -e^{2u} f^t + e^{2u} h^t (N^t_1 + N^t_2), \quad f^t_{zz} = 2u_z f^t_z + (\xi + c) N^t_1 + \xi N^t_2. \tag{91}$$

Hence the mean curvature vector of f^t is given by $\vec{H}_t = h^t(N^t_1 + N^t_2)$ and so f^t is marginally trapped. Also from (91) we see that

$$\langle f^t_{zz}, f^t_{zz} \rangle = (\xi + c)^2 - \xi^2 = 2\xi c + c^2 = \langle f_{zz}, f_{zz} \rangle. \quad \forall t \in R,$$

hence f^t is non-isotropic. On the other hand F^t is adapted to f^t since $F^t_z = F^t A^t$. From this equation we extract

$$\partial_z N^t_1 = -a^t_1 F^t_1 - ib^t_1 F^t_2, \quad \partial_z N^t_2 = a^t_2 F^t_1 + ib^t_2 F^t_2,$$

which implies that f^t has flat normal bundle for every t and also that $\{N^t_1, N^t_2\}$ is a parallel orthonormal frame with respect to the normal connection $\nabla^\perp_t$ of $\nu(f^t)$.

Equation (41) relating the conformal invariants and the δ differential of f reads

$$\kappa_{\bar{z}\bar{z}} + \frac{\bar{s}}{2}\kappa = ch\kappa, \quad c \in \mathbb{R}^\times, \quad \delta = chdz^2. \tag{92}$$

The deformation family f^t obtained above is locally defined on Σ and is related to (82) hence we call $f \mapsto f^t$ *the Calapso-Bianchi transformation* of the marginally trapped surface f.

Since $(s_t)_{\bar{z}} = s_{\bar{z}}$, then by Theorem 3.1 κ, s_t determine a unique (up to Moebius transformations of the sphere) conformally immersed isothermic surface $G^t : \Sigma \to \mathbb{S}^3$. Since for $t = 0$ we recover s in (82), G^t is the associated family of G or the T-transform of the isothermic surface $G : \Sigma \to \mathbb{S}^3$. We claim that G^t is the null Gauss map of f^t. In fact, from (91) it follows that $q = cdz^2$ is the Hopf differential of f^t. Since f^t induce the same conformal metric for all t, then θ in formula (32) must be an integer multiple of 2π, and so $\kappa = \frac{e^u}{\sqrt{2}}$ is the (common) normal Hopf differential of the null Gauss map of all f^t. Inserting (82) into (92) yields,

$$\kappa_{\bar{z}\bar{z}} + \frac{\overline{s_t}}{2}\kappa = c\left(h + \frac{t}{2c}\right) = ch^t\kappa, \quad \delta_t = ch^t dz^2, \tag{93}$$

where $\delta_t = ch^t dz^2$ is just the delta differential of f^t. Thus the above equation is the evolution of (92) and so κ, s_t are the conformal invariants of the null Gauss map of f^t. Thus G^t has conformal invariants κ, s_t and so it coincides up to a Moebius transformation of $\mathbb{S}^3$ with the null Gauss map of f^t which is isothermic since κ is real.

The transformation $f \mapsto f^t$ also preserves marginally trapped surfaces which are isothermic or have parallel second fundamental form. For instance if f is isothermic then for each $x \in \Sigma$ there is a local coordinate z for which $\Omega = \xi_1 N_1 dz^2 + \xi_2 N_2 dz^2$ is real valued, that is ξ_1, ξ_2 are real valued. Thus by Ricci's equation f has flat normal bundle and so the Hopf differential q is holomorphic by Lemma 4.2 and so $q = cdz^2$ for a non-zero real constant c, with $\xi_1 - \xi_2 = c$. Hence the function ξ in (83) satisfying $\xi_1 = \xi + c$, $\xi_2 = \xi$ must be also real valued. Thus from (85) it follows that the normal vector Hopf differential Ω^t of f^t is also real valued in the same coordinate z, which shows that f^t is isothermic for any $t \in \mathbb{R}$.

On the other hand if f has non-zero parallel mean curvature vector then it has flat normal bundle by Rahim Elghanmi (1996). Thus there is a local positive $\nabla^\perp$-parallel orthonormal frame $\{N_1, N_2\} \subset \Gamma(\nu(f))$ such that $0 = \nabla^\perp_{\partial_z} \vec{H} = h_z(N_1 + N_2)$, thus h is constant. Since h_t is defined by (86) it satisfies $(h^t)_z = h_z$, then $(h^t)_z = 0$ for all $t \in \mathbb{R}$ which shows that f^t has (non-zero) parallel mean curvature vector for any $t \in \mathbb{R}$. We summarize our discussion in the following

Theorem 5.1 *Let $f : \Sigma \to \mathbb{S}^4_1$ be a non-isotropic conformal marginally trapped immersion with flat normal bundle. Let $f^t : \tilde{\Sigma} \to \mathbb{S}^4_1$ be the Calapso-Bianchi deformation family of f obtained by integration of (89). Then on Σ each f^t is locally defined conformal non-isotropic marginally trapped immersion with flat normal bundle whose null Gauss map G^t is isothermic for any $t \in \mathbb{R}$.*

Moreover, the transformation $f \mapsto f^t$ preserves isothermic surfaces and surfaces with non-zero parallel mean curvature vector.

5.3 An extended deformation

Non-isotropic marginally trapped conformal immersed surfaces in $\mathbb{S}^4_1$ with non-zero parallel mean curvature vector have flat normal bundle (Rahim Elghanmi 1996) and have isothermic and constrained Willmore null Gauss maps into $\mathbb{S}^3$ by Theorem 4.2. In the previous sections we considered two different one parameter deformations for such surfaces, namely $f^\lambda, \lambda \in \mathbb{S}^1$ and $f^t, t \in \mathbb{R}$. Motivated by Burstall et al. (2002) we show that it is possible to unify both deformations by defining an (extended) action of $\mathbb{C} - \{0\}$ on the set of non-isotropic marginally trapped surfaces with non-zero parallel mean curvature vector.

Let $f : \Sigma \to \mathbb{S}^4_1$ be a non-isotropic conformally immersed marginally trapped surface with non-zero parallel mean curvature vector and κ, s be the conformal invariants of f, δ-differential $\delta = chdz^2$, with $h = const \neq 0$ and quadratic Hopf differential $q = cdz^2$, for real constant $c \neq 0$.

We extend the symmetry (79) for $\lambda \in \mathbb{C} - \{0\}$ by defining

$$\kappa_\lambda = |\lambda|^2 \lambda^{-2} \kappa, \quad s_\lambda = s + 2(\lambda^{-2} - 1)ch, \quad \delta_\lambda = \lambda^{-2}\delta. \tag{94}$$

Thus for $|\lambda| = 1$ above we recover (79). Moreover, since $ch\kappa$ is real, a straightforward calculation shows that $\kappa_\lambda, s_\lambda, \delta_\lambda$ above satisfy (41) and the conformal Gauss and Codazzi's equation (17) for every $\lambda \in \mathbb{C} - \{0\}$. Thus $\kappa_\lambda, s_\lambda, \delta_\lambda$ determine the extended associated family f^λ which for $|\lambda| = 1$ restricts to the associated family obtained in the previous section.

We briefly describe below the deformation given in (94) to a non-isotropic marginally trapped torus in $\mathbb{S}^4_1$ with non-zero parallel mean curvature. The image of the null Gauss map in this case is an isothermic constrained Willmore torus in $\mathbb{S}^3$ and by a result of Richter (1997) (see also Burstall et al. 2002) it can be immersed as a surface of constant mean curvature in some riemannian space form.

Let $f : \Sigma \to \mathbb{S}^4_1$ be a conformal non-isotropic marginally trapped immersion with non-zero parallel mean curvature then $Q = \langle f_{zz}, f_{zz}\rangle dz^4$ is holomorphic and non-zero. Away from the isolated zeros of Q it is possible to choose a local coordinate z such that $Q = dz^4$, or $\langle f_{zz}, f_{zz}\rangle = 1$. When $\Sigma = T^2$ is a 2-torus then Q has no zeros at all (otherwise Q would be identically zero). Thus $Q = dz^4$, where z is a global coordinate on the universal covering $\mathbb{C}$ of T^2 which determines a bi-holomorphism $T^2 \cong \mathbb{C}/\Gamma$ for some lattice $\Gamma_0 \subset \mathbb{C}$. We choose a positively oriented orthonormal lorentzian frame $\{N_1, N_2\} \subset \Gamma(\nu(f))$ such that

$$f_{zz} = 2u_z f_z + \cosh(C)N_1 + \sinh(C)N_2,$$

where $C = \rho + i\Theta$ is a complex function. The new positively oriented lorentzian frame $\{N_1', N_2'\}$ given by

$$N_1' = \cosh(\rho)N_1 + \sinh(\rho)N_2,$$
$$N_2' = \sinh(\rho)N_1 + \cosh(\rho)N_2,$$

has structure function $\sigma' = 0$ and so $\{N_1', N_2'\}$ is $\nabla^\perp$-parallel along f. Also since

$$f_{zz} = 2u_z f_z + \cos(\Theta)N_1' + i\sin(\Theta)N_2',$$

then Ricci's equation now becomes $0 = \cos(\Theta)\sin(\Theta)$, of which $\Theta = 0$ is a solution. For simplicity we drop the primes and keep denoting by $\{N_1, N_2\}$ this new $\nabla^\perp$-parallel normal frame. The structure equations of f become

$$\begin{aligned} f_{zz} &= 2u_z f_z + N_1,\\ f_{\bar{z}z} &= -e^{2u} f + e^{2u}h(N_1 + N_2),\\ \partial_z N_1 &= -hf_z - e^{-2u} f_{\bar{z}},\\ \partial_z N_2 &= hf_z, 0 \neq h = const. \end{aligned} \tag{95}$$

with compatibility given by the Sinh–Gordon equation $2u_{\bar{z}z} = -e^{2u} + e^{-2u}$, of which $u : \mathbb{C} \to \mathbb{R}$ is a doubly periodic solution with respect to the lattice $\Gamma_0 \subset \mathbb{C}$. Solutions to the Sinh–Gordon equation are obtained by applying the finite-gap integration method from theta functions defined on auxiliary hyperelliptic Riemann surfaces which arise from inverse scattering theory (Bobenko 1991).

Since the mean curvature vector is lightlike and non-zero the surface f cannot lie in a copy of $\mathbb{S}^3$, $\mathbb{H}^3$, $\mathbb{S}^3_1$ immersed as a totally geodesic hypersurface into $\mathbb{S}^4_1$. Moreover, $(N_1 + N_2)_z = -e^{-2u} f_{\bar{z}}$ implies that f cannot lie in any singular hypersurface of $\mathbb{S}^4_1$. From (95) the Hopf differential of f is given by $q = dz^2$, hence θ must be an integer multiple of 2π in (32) and so $\kappa = \frac{e^u}{\sqrt{2}}$ which says that $G : T^2 \to \mathbb{S}^3$ is an isothermic and constrained Willmore surface since h is a non-zero constant. The fundamental equation (41) becomes $\kappa_{\bar{z}\bar{z}} + \frac{\bar{s}}{2}\kappa = h\kappa, \quad \delta = hdz^2$.

We see from (94) that $\kappa_\lambda = |\lambda|^2\lambda^{-2}\frac{e^u}{\sqrt{2}}$. Also from (33) it follows that the f^λ-induced metric has conformal parameter u for every $\lambda \in \mathbb{C} - \{0\}$, thus all the surfaces in the extended family have the same induced metric. Using formula (32) we obtain the Hopf quadratic differential of f^λ:

$$q_\lambda = \lambda^{-2}|\lambda|^2 dz^2. \tag{96}$$

Thus since $\delta_\lambda = h_\lambda q_\lambda = \lambda^{-2}\delta = \lambda^{-2}hdz^2$, then the δ-differential of f^λ is given by $\delta_\lambda = \lambda^{-2}|\lambda|^2(\frac{h}{|\lambda|^2})dz^2$. Thus the marginally trapped torus f^λ has mean curvature function $h_\lambda = \frac{h}{|\lambda|^2}$ which is a non-zero constant since λ does not depend on z. Hence f^λ has non-zero parallel mean curvature vector and so its null Gauss map G^λ is constrained Willmore. In the new (rotated) coordinate $w := \frac{|\lambda|}{\lambda}z, \quad \kappa_\lambda$ is real with respect to w since $\kappa_\lambda dz^2 = \kappa dw^2$ and $\delta_\lambda = \frac{h}{|\lambda|^2}dw^2$, hence G^λ is isothermic for every $\lambda \in \mathbb{C} - \{0\}$. Note that for $t = 2h(\frac{1}{|\lambda|^2} - 1)$ we recover the Calapso-Bianchi transformation f^t of the marginally trapped torus f.

The structure equations of f^λ in the extended frame $F^\lambda = (f^\lambda, f^\lambda_z, f^\lambda_{\bar{z}}, N^\lambda_1, N^\lambda_2)$, $\lambda \in \mathbb{C} - \{0\}$, are thus given by

$$\begin{aligned} f^\lambda_{zz} &= 2u_z f^\lambda_z + \lambda^{-2}|\lambda|^2 N^\lambda_1, \\ f^\lambda_{\bar{z}z} &= -|\lambda|^4 e^{2u} f^\lambda + |\lambda|^4 e^{2u}\frac{h}{|\lambda|^2}(N^\lambda_1 + N^\lambda_2), \\ \partial_z N^\lambda_1 &= -\frac{h}{|\lambda|^2} f^\lambda_z - |\lambda|^{-4}e^{-2u} f^\lambda_{\bar{z}}, \\ \partial_z N^\lambda_2 &= \frac{h}{|\lambda|^2} f_z. \end{aligned} \tag{97}$$

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