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Spacelike surfaces in De Sitter 3-space and their twistor lifts

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ABSTRACT

We deal here with the geometry of the so-called twistor fibration $\mathcal{Z} \to \mathbb{S}_1^3$ over the De Sitter 3-space, where the total space $\mathcal{Z}$ is a five-dimensional reductive homogeneous space with two canonical invariant almost CR structures. Fixed the normal metric on $\mathcal{Z}$ we study the harmonic map equation for smooth maps of Riemann surfaces into $\mathcal{Z}$. A characterization of spacelike surfaces with harmonic twistor lifts to $\mathcal{Z}$ is obtained. Also it is shown that the harmonic map equation for twistor lifts can be formulated as the curvature vanishing of an $\mathbb{S}^1$-loop of connections i.e. harmonic twistor lifts exist within $\mathbb{S}^1$-families. Special harmonic maps such as holomorphic twistor lifts are also considered and some remarks concerning (compact) vacua of the twistor energy are given.

1. Introduction

J. Eells and S. Salamon [9] were able to obtain conformal harmonic maps of Riemann surfaces into a 3-dimensional Riemannian manifold N by projecting Cauchy–Riemann holomorphic maps with values in the unit tangent bundle T^1N. Replacing the Riemannian 3-manifold N by a Lorentzian 3-manifold L, it is then natural to look for analogous results. The "unit" tangent bundle $T^1L = \{x \in TL\colon\ \|x\|^2 = \pm 1\}$, of a Lorentzian 3-manifold L splits into the disjoint union of the timelike unit tangent bundle $T^1_{-1}L = \{x \in TL\colon\ \|x\|^2 = -1\}$ and the spacelike unit tangent bundle $T^1_{+1}L = \{x \in TL\colon\ \|x\|^2 = +1\}$. If L comes equipped with a global notion of time orientedness, then any connected spacelike surface in L has a uniquely defined future oriented timelike unit normal field $\mathbf{n}$ which may be viewed as a kind of Gauss lift of the surface to the subbundle $\mathcal{Z} \subset T^1_{-1}L$ consisting of all future oriented timelike unit tangent vectors of L. We call $\mathcal{Z}$ the twistor bundle of the Lorentzian 3-manifold L, by analogy with the construction of hyperbolic twistor spaces due to D. Blair, J. Davidov and O. Muskarov [6].

Our goal in this paper is to give a detailed account of the geometry of the twistor bundle $\mathcal{Z}$ constructed over De Sitter 3-space or pseudosphere $\mathbb{S}_1^3$. Also we intend to understand the geometry behind the harmonic map equation for maps from Riemann surfaces into $\mathcal{Z}$, when the normal metric is fixed on the target. In particular we characterize conformally immersed (hence spacelike) surfaces in $\mathbb{S}_1^3$ whose lifts are harmonic maps.

As a homogeneous manifold of the simple Lie group $SO_o(3,1)$ the twistor space $\mathcal{Z}$ is a (non-compact) reductive quotient equipped with an invariant horizontal distribution $\mathfrak{h} \subset T\mathcal{Z}$ which is just the orthogonal complement with respect to the normal metric to the vertical distribution of the principal bundle $\mathcal{Z} \to G_2^+(\mathbb{R}_1^4)$, where the last is the Grassmann manifold

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of oriented spacelike subspaces of $\mathbb{R}^4_1$. The distribution $\mathfrak{h} \subset T\mathcal{Z}$ supports two invariant almost complex structures which arise from the natural complex structure on the fibers of $\mathcal{Z} \to \mathbb{S}^3_1$ (diffeomorphic to the hyperbolic 2-space $\mathbb{H}^2$), and the almost complex structure naturally attached to every point of $\mathcal{Z}$. These almost complex structures give rise to corresponding CR structures on $\mathcal{Z}$. Since the horizontal distribution $\mathfrak{h}$ is contact, its integral manifolds have dimension at most two. In fact, twistor lifts of conformally immersed Riemann surfaces provide examples of Legendrian manifolds (maximal integral manifolds) of the distribution $\mathfrak{h}$. We consider the energy of smooth maps $\phi : M \to \mathcal{Z}$ determined by the normal metric on $\mathcal{Z}$. In this context we investigate those lifts which are critical points of the energy, i.e. which satisfy the harmonic map equation. Our main result is Theorem 5.2 which gives a characterization conformally immersed surfaces in $\mathbb{S}^3_1$ with harmonic twistor lift. Also in Theorem 6.1 we show that the harmonic map equation for twistor lifts is a completely integrable system, i.e. harmonic twistor lifts exist within $\mathbb{S}^1$-families.

The paper is organized as follows. In the first section we derive the basic structure equations of conformally immersed Riemann surfaces in $\mathbb{S}^3_1$. The second section is devoted to study the geometry of the twistor fibration $\mathcal{Z} \to \mathbb{S}^3_1$. We introduce the horizontal distribution $\mathfrak{h} \subset T\mathcal{Z}$ and define two invariant almost complex structures J', J'' on $\mathcal{Z}$. In the third section we derive the harmonic map equation for smooth maps of Riemann surfaces with values in $\mathcal{Z}$ in terms of the Maurer–Cartan one form β of the reductive space $\mathcal{Z}$. In Section 4 we characterize spacelike surfaces with harmonic twistor lifts. In Section 5 we deal with one (complex) parameter deformations of harmonic twistor lifts. Although $\mathcal{Z}$ is not a symmetric space, we show that the harmonic map equation for twistor lifts can be formulated as a loop of flat connections, a characteristic property of integrable systems (see [4] for instance). We also consider special twistor lifts such as holomorphic ones. However, we have not dealt here with specific calculations using loop groups techniques to produce examples of harmonic lifts. This will be considered in another paper. Finally in the last section we compute the energy of twistor lifts and establish a relationship with the Willmore energy. Some remarks are given concerning compact vacua of the twistor energy.

2. Spacelike surfaces in $\mathbb{S}^3_1$

Denote by $\mathbb{R}^4_1$ the real 4-space $\mathbb{R}^4$ with coordinates (x_1, x_2, x_3, x_4) equipped with the Lorentz metric

$$\langle .\, , . \rangle = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2.$$

De Sitter 3-space is defined as the unit sphere in $\mathbb{R}^4_1$

$$\mathbb{S}^3_1 = \{x \in \mathbb{R}^4_1 \colon \langle x, x \rangle = 1\},$$

on which the ambient Lorentz metric induces a pseudometric $\langle .\, , . \rangle$ with signature $(+ + -)$, and so it becomes a Lorentz 3-manifold with constant curvature one.

Let $I_{3,1} = \mathrm{diag}(1, 1, 1, -1)$, then the simple Lie group $SO_o(3, 1) = \{A \in Gl_4(\mathbb{R})\colon\ A^t I_{3,1} A = I_{3,1},\ a_{44} > 0\}$ acts transitively on $\mathbb{S}^3_1$ by isometries. A global time orientation of $\mathbb{S}^3_1$ is obtained by declaring a timelike vector $X \in T_x\mathbb{S}^3_1$ to be *future-pointing* if $\langle X, V_x \rangle < 0$, where V is the unit timelike Killing vector field V on $\mathbb{S}^3_1$ given by

$$V_x = \frac{d}{dt}\bigg|_{t=0} \exp(tX_0).x, \quad x \in \mathbb{S}^3_1,$$

and

$$X_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(3, 1).$$

It is easily seen that a timelike vector $X \in T_x\mathbb{S}^3_1$ is future pointing if and only if after parallel translation to the origin of $\mathbb{R}^4_1$ it satisfies $X_4 > 0$.

An immersion $f : M \to \mathbb{S}^3_1$ of a Riemann surface is conformal if $\langle f_z, f_z \rangle^c = 0$, for every local complex coordinate $z = x + iy$ on M, where

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

are the complex partial derivatives and $\langle\, , \rangle^c$ is the complex bilinear extension of the Lorentz metric to $\mathbb{C}^4$:

$$\langle z, w \rangle^c = z_1 w_1 + z_2 w_2 + z_3 w_3 - z_4 w_4.$$

Thus f is conformal if and only if on any local complex coordinate $z = x + iy$ it satisfies

$$\langle f_x, f_y \rangle = 0, \qquad \|f_x\|^2 = \|f_y\|^2. \tag{1}$$

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In particular (1) implies that every conformal immersion $f : M \to \mathbb{S}^3_1$ is spacelike, i.e. the induced metric $g = f^*\langle\, ,\rangle$ is positive definite or Riemannian on M.

Let $f : M \to \mathbb{S}^3_1$ be a conformal immersion of a connected Riemann surface M and $\mathbf{n} : M \to T\mathbb{S}^3_1$ a smooth future-pointing unit timelike vector field along f which is normal to the immersed surface at each point. Such vector field exists by the time orientation of $\mathbb{S}^3_1$. We fix on M the induced Riemannian metric $g = f^*\langle\, ,\rangle$ so that $f : (M, g) \to \mathbb{S}^3_1$ is a spacelike isometric immersion. The 2*nd Fundamental form* of the space-like surface $f : M \to \mathbb{S}^3_1$ is given by

$$II = -\langle df, d\mathbf{n}\rangle.$$

On any local chart $(U, z = x + iy)$ of M we introduce a conformal parameter u defined by $\langle \partial f, \partial f\rangle = e^{2u}$, so that $g|_U = 2e^{2u}(dx^2 + dy^2)$. The *Mean curvature* of f is defined by $H = \frac{1}{2}\, trace\, II$, which in terms of f and u is given by

$$H = -e^{-2u}\langle f_{\bar{z}z}, \mathbf{n}\rangle.$$

Since f is conformal we have

$$2\langle f_{\bar{z}z}, f_z\rangle^c = \frac{\partial}{\partial z}\langle f_z, f_z\rangle^c = 0,$$
$$2\langle f_{\bar{z}z}, f_{\bar{z}}\rangle^c = \frac{\partial}{\partial z}\langle f_{\bar{z}}, f_{\bar{z}}\rangle^c = 0,$$

hence $f_{\bar{z}z}$ has no tangential component. We obtain the structural equations of f:

$$\begin{aligned} f_{\bar{z}z} &= -e^{2u} f + e^{2u} H.\mathbf{n}, && \text{(i)}\\ f_{zz} &= 2u_z.f_z + \xi.\mathbf{n}, && \text{(ii)}\\ \mathbf{n}_z &= H.f_z + e^{-2u}\xi.f_{\bar{z}}, && \text{(iii)} \end{aligned}$$

where $q := \xi\, dz \otimes dz = -\langle f_{zz}, \mathbf{n}\rangle^c dz^2$ is the Hopf quadratic complex differential. Zeros of q are the umbilic points of M. We say that f is isotropic or totally umbilic if and only if $\xi \equiv 0$. If $H \equiv 0$ then the conformal immersion f is harmonic or maximal. In this case f satisfies

$$f_{\bar{z}z} = -\langle f_z, f_{\bar{z}}\rangle^c f, \tag{2}$$

where $\langle f_z, f_{\bar{z}}\rangle^c = e^{2u}$. Away from umbilic points of f, from Eq. (ii) we obtain

$$\mathbf{n} = \frac{1}{\xi}.(2u_z f_z - f_{zz}), \tag{3}$$

which allows us to recover the normal vector field from the immersion.

The compatibility conditions of the structure equations are the Gauss–Codazzi equations:

$$\begin{aligned} 2u_{\bar{z}z} &= (H^2 - 1)e^{2u} - |\xi|^2 e^{-2u} && \text{(Gauss)},\\ \xi_{\bar{z}} &= e^{2u} H_z && \text{(Codazzi)}. \end{aligned} \tag{4}$$

Conversely, it is known that any solution of these equations defines a surface in $\mathbb{S}^3_1$ up to an isometry. From Codazzi's equation a surface has contant mean curvature H if and only if ξ is holomorphic. Hence $H = const$ and $\xi \not\equiv 0$, then the umbilic points are isolated.

For a conformal immersion $f : M \to \mathbb{S}^3_1$, we consider the induced metric $g = f^*\langle\, ,\rangle$ on M, hence $f : (M, g) \to \mathbb{S}^3_1$ is an isometric space-like immersion. In terms of the conformal parameter u, the induced metric is given by $g = 2e^{2u} dz \otimes d\bar{z}$, and the Gaussian curvature of the surface (M, g) is just the curvature of the induced metric and is given by

$$K = -\Delta_g u = -2e^{-2u} u_{\bar{z}z},$$

where Δ_g is the Laplace operator on M determined by g. In complex coordinates $\Delta_g = 2e^{-2u}\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}$. Thus Gauss equation becomes

$$K = 1 - H^2 + |\xi|^2 e^{-4u}. \tag{5}$$

Here $\|q\| = |\xi|^2 e^{-4u}$ is the intrinsic norm of the Hopf differential. If λ_1, λ_2 are the principal curvatures of the immersed surface, then it follows that

$$|\xi|^2 e^{-4u} = \frac{1}{4}(\lambda_1 - \lambda_2)^2, \tag{6}$$

so that Gauss equation reads

$$K + H^2 - 1 = \frac{1}{4}(\lambda_1 - \lambda_2)^2. \tag{7}$$

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3. The twistor bundle of $\mathbb{S}_1^3$ and its geometry

By definition the fiber G_v of the Gauss bundle $G \to \mathbb{S}_1^3$ over a point $v \in \mathbb{S}_1^3$ is the totality of 2-planes passing through the origin of $T_v\mathbb{S}_1^3$. If one is interested only in the geometry of spacelike surfaces in $\mathbb{S}_1^3$, the Gauss bundle is too big. Restricting the fibers of G by allowing only spacelike subspaces, one obtains a new bundle. We define the twistor bundle $\mathcal{Z} \subset G$ of $\mathbb{S}_1^3$ by demanding that the fiber $\mathcal{Z}_v$ over a point $v \in \mathbb{S}_1^3$ be the set of spacelike 2-planes in $T_v\mathbb{S}_1^3$. A spacelike 2-plane $V \subset T_v\mathbb{S}_1^3$ has two timelike unit normal vectors of which only one is (according to our definition) future pointing and we choose it to fix the desired orientation on V. Hence any $V \in \mathcal{Z}_v$ determines and is determined by a unit timelike future pointing vector $w \in T_v\mathbb{S}_1^3$ by requiring $w^\perp = V \subset T_v\mathbb{S}_1^3$. Translating w and V to the origin of $\mathbb{R}_1^4$, they satisfy $\langle w, w\rangle = -1$, $\langle v, w\rangle = 0$ with $w_4 > 0$ and $V = [v \wedge w]^\perp$. Note that w defines a point in the upper half $\mathbb{H}_+^3$ of the hyperboloid $\{x \in \mathbb{R}_1^4\colon\ \langle x, x\rangle = -1\}$, which is the unbounded realization of the three-dimensional real hyperbolic space. Hence the total space of the twistor bundle of $\mathbb{S}_1^3$ is just

$$\mathcal{Z} = \big\{(v, w) \in \mathbb{S}_1^3 \times \mathbb{H}_+^3 \subset \mathbb{R}_1^4 \times \mathbb{R}_1^4\colon\ \langle v, w\rangle = 0\big\}, \tag{8}$$

where the projection map $\pi : \mathcal{Z} \to \mathbb{S}_1^3$ is simply $\pi(v, w) = v$. The fiber $\mathcal{Z}_v = \pi^{-1}(v)$ over $v \in \mathbb{S}_1^3$ identifies with $v^\perp \cap \mathbb{H}_+^3$, which is a copy of hyperbolic 2-space $\mathbb{H}^2$ totally geodesic immersed in $\mathbb{H}_+^3$, hence a complex manifold.

Note that a second fibration $\pi'' : \mathcal{Z} \to \mathbb{H}_+^3$ is obtained by projection on the second factor, $\pi'' : (v, w) \mapsto w$. It is clear that the fiber of an element $w \in \mathbb{H}_+^3$ is the 2-sphere $w^\perp \cap \mathbb{S}_1^3$.

Remark 3.1. Any $w \in \mathcal{Z}_v$ determines the oriented spacelike 2-plane $V = [v \wedge w]^\perp \subset T_v\mathbb{S}_1^3$ to which there is associated the rotation $J_w : V \to V$ of angle $\frac{\pi}{2}$ compatible with the orientation on V. So that identifying

$$w \leftrightarrow [v \wedge w]^\perp \leftrightarrow J_w,$$

we may think of $\mathcal{Z}_v \equiv \mathbb{H}^2$ as the set of all oriented spacelike planes in $T_v\mathbb{S}_1^3$ and also, as the set of all oriented rotations in $T_v\mathbb{S}_1^3$. One may view a section of $\mathcal{Z}$ as a field of rotations $\{v \mapsto J_v\}$, or equivalently as a distribution $\mathcal{D}$ of oriented spacelike 2-planes in $\mathbb{S}_1^3$.

Let M be a connected Riemann surface and $f : M \to \mathbb{S}_1^3$ a conformal immersion. It is not hard to show that there exists a uniquely defined future-oriented unit normal vector field $\hat{f}$ along f satisfying

$$\hat{f}(x) \in T_{f(x)}\mathbb{S}_1^3, \qquad \hat{f}(x) \perp df_x(T_xM), \quad \forall x \in M.$$

Thus the field $\hat{f}$ determines a unique smooth map $\mathbf{n} : M \to \mathbb{H}_+^3$ such that

$$\hat{f}(x) = \big(f(x), \mathbf{n}(x)\big) \in \mathcal{Z}, \quad x \in M. \tag{9}$$

We call $\hat{f} : M \to \mathcal{Z}$ the *twistor lift* of the conformal immersion f, and $\mathbf{n} : M \to \mathbb{H}_1^3$ its *normal Gauss map*.

Since $\mathcal{Z}$ is a submanifold of $\mathbb{R}_1^4 \times \mathbb{R}_1^4$, the tangent space of $\mathcal{Z}$ at (v, w) is given by

$$T_{(v,w)}\mathcal{Z} = \big\{(x, y) \in \mathbb{R}_1^4 \times \mathbb{R}_1^4\colon\ \langle x, v\rangle = \langle y, w\rangle = 0,\ \langle x, w\rangle + \langle v, y\rangle = 0\big\}.$$

Hence fixed the base point $o = (e_1, e_4) \in \mathcal{Z}$ then $(x, y) \in T_o\mathcal{Z}$ if and only if

$$x = (0, x_2, x_3, x_4)^T, \qquad y = (x_4, y_2, y_3, 0)^T.$$

Then we may identify

$$\mathfrak{p} := T_o\mathcal{Z} \ni (x, y) \equiv \begin{pmatrix} 0 & x_2 & x_3 & x_4 \\ -x_2 & 0 & 0 & y_2 \\ -x_3 & 0 & 0 & y_3 \\ x_4 & y_2 & y_3 & 0 \end{pmatrix} \in \mathfrak{so}(3, 1). \tag{10}$$

On the other hand the transitive left action of $SO_o(3, 1)$ on $\mathcal{Z}$ given by $g.(v, w) = (g.v, g.w)$ allows to identify $\mathcal{Z}$ with the quotient $SO_o(3, 1)/K$, where

$$K = \big\{\mathrm{diag}(1, A, 1),\ A \in SO(2)\big\}, \tag{11}$$

is the isotropy subgroup of the base point $o \in \mathcal{Z}$.

Decompose $\mathfrak{so}(3, 1) = \mathfrak{k} \oplus \mathfrak{p}$, where

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$$\mathfrak{k} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ a \in R \right\},$$

is the Lie algebra of $K \simeq SO(2)$. Then the decomposition is reductive since $\mathfrak{p} \equiv T_o\mathcal{Z}$ defined above is $Ad(K)$-invariant. From now on we choose the $Ad(K)$-invariant inner product $\langle .,. \rangle$ on $\mathfrak{p} \equiv T_o\mathcal{Z}$ defined by

$$\langle A, B\rangle = -\frac{1}{2} trace(A.B), \quad A, B \in \mathfrak{p}. \tag{12}$$

Note that $\|A\|^2 = x^2 + y^2 - c^2 - z^2 - w^2$, $\forall A \in \mathfrak{p}$, hence (12) gives rise to an $SO_o(3,1)$-invariant pseudo-metric on $\mathcal{Z}$ of signature $(++---)$ denoted also by $\langle .,. \rangle$, the so-called *normal metric*. Since $\langle .,. \rangle$ is the restriction of (a multiple of) the Killing form of $\mathfrak{so}(3,1)$ to $\mathfrak{p} \times \mathfrak{p}$, $(\mathcal{Z}, \langle .,. \rangle)$ is naturally reductive. In this case the natural projection $SO_o(3,1) \to \mathcal{Z}$ is a (pseudo) Riemannian submersion, in which the bi-invariant (pseudo) metric induced by the Killing form is considered on $SO_o(3,1)$. From now on we shall consider on $\mathcal{Z}$ only the normal metric.

Remark 3.2. Another remarkable $SO_o(3,1)$-invariant metric is the one that makes the inclusion $\mathcal{Z} \subset \mathbb{R}^4_1 \times \mathbb{R}^4_1$ an isometric immersion.

Given $(v, w) \in \mathcal{Z}$ the oriented space-like 2-plane $V = [v \wedge w]^\perp$ defines a point of the Grassmannian $G^+_2(\mathbb{R}^4_1)$ of all oriented spacelike planes in $\mathbb{R}^4_1$. Defining the projection map $\pi' : \mathcal{Z} \to G^+_2(\mathbb{R}^4_1)$, by

$$\pi'(v, w) = [v \wedge w]^\perp,$$

we obtain an $SO_o(1,1)$-principal bundle $\pi' : \mathcal{Z} \to G^+_2(\mathbb{R}^4_1)$ where the right action of $SO_o(1,1)$ on the total space $\mathcal{Z}$ is given by

$$(v, w).\begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} = \big(\cosh(t)v + \sinh(t)w, \sinh(t)v + \cosh(t)w\big). \tag{13}$$

Pick a point $V \in G^+_2(R^4_1)$ and let $\{v, w\}$ be an oriented basis of $V^\perp$ with $\|v\|^2 = -\|w\|^2 = 1$, $\langle v, w\rangle = 0$ and $w_4 > 0$, so that $(v, w) \in \mathcal{Z}$ and $V = [v \wedge w]^\perp$. Define the curve $\tau(t) = \cosh(t)v + \sinh(t)w \subset V^\perp = [v \wedge w]$, then $\gamma(t) = (\tau(t), \tau'(t)) \in \mathcal{Z}$ for all $t \in \mathbb{R}$ and $\gamma(0) = (v, w)$. We conclude that

$$\pi'^{-1}(V) = \big\{\big(\tau(t), \dot\tau(t)\big) \colon t \in \mathbb{R}\big\}.$$

In other words (13) shows that the π'-fibre through $(v, w) \in \mathcal{Z}$ is just the (right) orbit

$$(v, w).SO_o(1,1) = \big\{(v, w).\exp(tZ_0) \colon t \in \mathbb{R}\big\}, \quad Z_0 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{so}(1,1).$$

Note that for any $(v, w) \in \mathcal{Z}$ we have

$$\frac{d}{dt}\bigg|_{t=0} (v, w).\exp(tZ_0) = \frac{d}{dt}\bigg|_{t=0} \big(\tau(t), \dot\tau(t)\big) = (w, v) \in T_{(v,w)}\mathcal{Z}.$$

This suggests defining the characteristic (or Hopf) vector field h on $\mathcal{Z}$ by

$$h_{(v,w)} = (w, v) \in T_{(v,w)}\mathcal{Z}. \tag{14}$$

Hence h is a unit timelike vector field, i.e. $\|h\|^2 = -1$ on $\mathcal{Z}$. In this way $\pi' : \mathcal{Z} \to G^+_2(\mathbb{R}^4_1)$ may be viewed as an analogous the usual Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^2$.

Next we define the horizontal distribution $\mathfrak{h} \subset T\mathcal{Z}$ as the orthogonal complement with respect to the normal metric of the vertical distribution namely,

$$\mathfrak{h}_{(v,w)} = h^\perp_{(u,v)} \subset T_{(v,w)}\mathcal{Z}, \quad \forall (v, w) \in \mathcal{Z}, \tag{15}$$

thus $\mathfrak{h} \subset T\mathcal{Z}$ is the complementary subbundle of the fibres of π' and we decompose

$$T_{(v,w)}\mathcal{Z} = \mathfrak{h}_{(v,w)} \overset{\perp}{\oplus} \mathbb{R}(w, v), \tag{16}$$

where $Ker\, d\pi'_{(v,w)} = \mathbb{R}h_{(v,w)} = \mathbb{R}(w, v)$. It is not difficult to verify that $\mathfrak{h}$ defines a connection on the principal bundle $SO_o(1,1) \to \mathcal{Z} \to G^+_2(R^4_1)$. This is an example of sub-semi-Riemannian geometry in which the metric on the distribution $\mathfrak{h}$ is non-definite.

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Lemma 3.1. *(See [10].) For every $(v, w) \in \mathcal{Z}$ the subspaces $Ker\, d\pi_{(v,w)}$ and $Ker\, d\pi'_{(v,w)}$ are orthogonal w.r.t. the normal metric (12).*

From the preceding lemma and (16) it follows that $Ker\, d\pi_{(v,w)} \subset \mathfrak{h}_{(v,w)}$. Thus we have the orthogonal decomposition

$$\mathfrak{h}_{(v,w)} = Ker\, d\pi_{(v,w)} \overset{\perp}{\oplus} L_{(v,w)}, \tag{17}$$

in which the subspace $L_{(v,w)}$ is the horizontal lift via $d\pi$ of the oriented spacelike 2-plane $V = [v \wedge w]^{\perp} \subset T_v\mathbb{S}^3_1$. Note that the normal metric (12) restricted to $\mathfrak{h}$ has signature $(++)$ on L, and $(--)$ on $Ker\, d\pi$. Hence the invariant metric on $\mathfrak{h}$ is neutral, i.e. has signature $(++--)$.

The geometry of the twistor space $\mathcal{Z}$ may be studied with the aid of the so-called Maurer–Cartan form β of $\mathcal{Z}$ introduced by Burstall and Rawnsley in [7] of which we give a brief account.

Let $\mathfrak{g} = \mathfrak{so}(3,1)$ and recall the reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Consider the surjective application $\xi_o : \mathfrak{g} \ni X \mapsto \frac{d}{dt}|_{t=0} \exp(tX).o \in T_o\mathcal{Z}$. It follows that ξ_o has kernel $\mathfrak{k}$ and restricts to an isomorphism $\mathfrak{p} \to T_o\mathcal{Z}$. Now form the associated vector bundle $[\mathfrak{p}] := SO_o(3,1) \times_K \mathfrak{p}$. Then the map

$$\big[(g, X)\big] \mapsto \frac{d}{dt}\bigg|_{t=0} \exp\big(t\, Ad(g)X\big)x = d\tau_g\left(\frac{d}{dt}\bigg|_{t=0} \exp(tX).o\right), \quad x = g.o,$$

establishes an isomorphism of the associated bundle $[\mathfrak{p}]$ and the tangent bundle $T\mathcal{Z}$, where τ_g is the isometry of $\mathcal{Z}$ sending $g'.o$ to $gg'.o$.

Since $\mathfrak{p}$ is an $Ad(K)$-invariant subspace of $\mathfrak{g}$, one has the inclusion $[\mathfrak{p}] \subset [\mathfrak{g}] := \mathcal{Z} \times \mathfrak{g}$, given by $[\mathfrak{p}] \ni [(g, X)] \mapsto (g.o, Ad(g)X) \in [\mathfrak{g}]$. Note that the fiber of $[\mathfrak{p}] \to \mathcal{Z}$ over the point $g.o$ identifies with $\{g.o\} \times Ad(g)\mathfrak{p} \subset [\mathfrak{g}]$. This shows that there exists an identification of $T\mathcal{Z}$ with a subbundle of the trivial bundle $[\mathfrak{g}]$. This inclusion may be viewed as an $\mathfrak{g}$-valued one-form on $\mathcal{Z}$ which will be denoted by β. Note that every $X \in \mathfrak{g}$ determines a flow on $\mathcal{Z}$ defined by $\varphi_t(x) = \exp(tX).x$, which is an isometry of $\mathcal{Z}$ for any $t \in \mathbb{R}$. The vector field of the flow is then a Killing field denoted by X^* which is given by

$$X^*_x = \frac{d}{dt}\bigg|_{t=0} \exp(tX)x, \quad \forall x \in \mathcal{Z}.$$

It is not difficult to show that

$$\beta_x\big(X^*_x\big) = Ad(g)\big[Ad\big(g^{-1}\big)(X)\big]_{\mathfrak{p}}, \quad \forall X \in \mathfrak{p},$$

where $x = g.o \in \mathcal{Z}$. In particular at $o \in \mathcal{Z}$ we have $\beta_o(X^*_o) = X$ for any $X \in \mathfrak{p}$. From this formula it follows the equivariance of β which is expressed by

$$\beta \circ d\tau_g = Ad(g)\beta, \quad \forall g \in SO_o(3,1). \tag{18}$$

For $x = g.o \in \mathcal{Z}$ the application $\xi_x : \mathfrak{g} \to T_x\mathcal{Z}$ such that $X \overset{\xi_x}{\longmapsto} X^*_x$, maps $\mathfrak{so}(3,1)$ onto $T_x\mathcal{Z}$, and restricts to an isomorphism $Ad(g)(\mathfrak{p}) \to T_x\mathcal{Z}$ whose inverse conicides with β_x. Note that ξ satisfies $d\tau_g \circ \xi_o(X) = \xi_{g.o}(Ad(g)X)$, which is equivalent to (18). More details and properties of the one form β and proofs can be found in [7].

Now recall the definition of the horizontal distribution $\mathfrak{h} \subset T\mathcal{Z}$ given in (15). At the basepoint $o = (e_1, e_4) \in \mathcal{Z}$ the subspace determined by the horizontal distribution identifies with

$$\mathcal{H} = \left\{ \begin{pmatrix} 0 & x & y & 0 \\ -x & 0 & 0 & z \\ -y & 0 & 0 & w \\ 0 & z & w & 0 \end{pmatrix},\ x, y, z, w \in \mathbb{R} \right\} \subset \mathfrak{p}, \tag{19}$$

which is an $Ad(K)$-invariant subspace of $\mathfrak{p}$.

The one-form β transfers the metric on the fibers of $T\mathcal{Z}$, the horizontal distribution $\mathfrak{h}$ to the fibers of $[\mathfrak{p}]$. From the definition of β above we get

$$\beta_{g.o}(\mathfrak{h}_{g.o}) = Ad(g)\mathcal{H} = [\mathcal{H}]_{g.o}, \quad \forall g \in SO_o(3,1). \tag{20}$$

Thus β identifies the horizontal distribution $\mathfrak{h} \subset T\mathcal{Z}$ with the subbundle $[\mathcal{H}] \subset [\mathfrak{p}]$.

3.1. Invariant almost complex structures on $\mathfrak{h}$

Let $L_{(v,w)} \subset \mathfrak{h}_{(v,w)}$ be the $d\pi$-horizontal lift of the oriented spacelike 2-plane $V = [v \wedge w]^{\perp} \subset T_v\mathbb{S}^3_1$. Denote by $J^L_{(v,w)} : L_{(v,w)} \to L_{(v,w)}$ the $d\pi$-horizontal lift of the positively oriented $\frac{\pi}{2}$-rotation on the spacelike 2-plane $V = [v \wedge w]^{\perp}$.

On the other hand let $J^V_{(v,w)}$ be the complex structure on the tangent space $Ker\, d\pi_{(v,w)}$ of the fibre of v (recall that the fibers of $\pi : \mathcal{Z} \to \mathbb{S}^3_1$ are hyperbolic 2-spaces, hence complex manifolds). Both structures together yield an almost complex structure J' on the distribution $\mathfrak{h} = L \oplus Ker\, d\pi$ which is defined by

$$J' = \begin{cases} J^L, & \text{on } L, \\ J^{\mathcal{V}}, & \text{on } Ker\, d\pi. \end{cases} \tag{21}$$

At the base point $o = (e_1, e_4)$ it is possible to describe explicitly the action of J. In fact

$$L_o = \left\{ \begin{pmatrix} 0 & x & y & 0 \\ -x & 0 & 0 & 0 \\ -y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}, \qquad Ker\, d\pi_o = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & w \\ 0 & z & w & 0 \end{pmatrix} \right\}.$$

On L_o the complex structure is obtained by lifting via $d\pi|_o$ the oriented rotation on $[e_2 \wedge e_3] = [e_1 \wedge e_4]^{\perp} \subset T_{e_1}\mathbb{S}^3_1$ given by

$$e_2 \mapsto e_3, \qquad e_3 \mapsto -e_2.$$

The complex structure $J^{\mathcal{V}}$ on the fibre

$$\mathcal{Z}_{e_1} = \pi^{-1}(e_1) \equiv \mathbb{H}^2 \equiv SO(2,1)/SO(2)$$

is given at $o \in \mathcal{Z}_{e_1}$ by

$$J^{\mathcal{V}}: \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & w \\ 0 & z & w & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -w \\ 0 & 0 & 0 & z \\ 0 & -w & z & 0 \end{pmatrix}.$$

According to the definition given before, the almost complex structure $J' : \mathcal{H} \to \mathcal{H}$ is given by

$$J' \begin{pmatrix} 0 & x & y & 0 \\ -x & 0 & 0 & z \\ -y & 0 & 0 & w \\ 0 & z & w & 0 \end{pmatrix} = \begin{pmatrix} 0 & -y & x & 0 \\ y & 0 & 0 & -w \\ -x & 0 & 0 & z \\ 0 & -w & z & 0 \end{pmatrix}. \tag{22}$$

The second almost complex structure J'' on $\mathfrak{h}$ is obtained by reversing the complex structure on the fibers, namely

$$J'' = \begin{cases} J^L, & \text{on } L, \\ -J^{\mathcal{V}}, & \text{on } Ker\, d\pi. \end{cases} \tag{23}$$

The action of J'' on the subspace $\mathcal{H} \subset \mathfrak{p}$ is given by

$$J'' \begin{pmatrix} 0 & x & y & 0 \\ -x & 0 & 0 & z \\ -y & 0 & 0 & w \\ 0 & z & w & 0 \end{pmatrix} = \begin{pmatrix} 0 & -y & x & 0 \\ y & 0 & 0 & w \\ -x & 0 & 0 & -z \\ 0 & w & -z & 0 \end{pmatrix}. \tag{24}$$

We summarize our discussion above in the following

Lemma 3.2. *Let J be either J' or J''. Then J commute with $\{Ad(x)|_{\mathcal{H}}: x \in K\}$ and is orthogonal, i.e.*

$$\langle JX, JY \rangle = \langle X, Y \rangle, \quad \forall X, Y \in \mathcal{H},$$

where $J \in \{J', J''\}$. Thus J' and J'' are $SO_o(3,1)$-invariant almost complex structures on $\mathfrak{h} = [\mathcal{H}] \subset T\mathcal{Z}$.

As a consequence of a theorem by LeBrun it can be shown that the almost CR structure $(\mathfrak{h} = [\mathcal{H}], J')$ on $\mathcal{Z}$ is integrable (see [11]).

4. Harmonic maps into $\mathcal{Z}$

Here we study the harmonic map equation for smooth maps $\phi : M \to \mathcal{Z}$ from a Riemann surface into the twistor bundle, on which we have fixed the normal metric $\langle .,. \rangle$. Let $\Omega \subset M$ be a relatively compact domain of M and define the twistor energy of ϕ over Ω by

$$E_{\Omega}(\phi) = \frac{1}{2} \int_{\Omega} \|d\phi\|^2\, dA, \tag{25}$$

where dA is the area form on M determined by a conformal metric g, and $\|d\phi\|^2$ is the Hilbert–Schmidt norm of $d\phi$ defined by $\|d\phi\|^2 = \sum_i \langle d\phi(e_i), d\phi(e_i) \rangle$ for any orthonormal frame $\{e_i\}$ on M. By definition ϕ is harmonic if it is an extreme of the energy functional $\phi \mapsto E_{\Omega}(\phi)$ over all relatively compact subdomains $\Omega \subset M$. Note that the energy may be negative since the metric on $\mathcal{Z}$ is indefinite.

Let us denote by ∇ the Levi-Civita connection on $\mathcal{Z}$ determined by the normal metric, and by ∇^{ϕ} the induced connection on the pullback bundle $\phi^{-1}T\mathcal{Z}$. By the formula of the first variation of the energy (see [8]) it follows that ϕ is harmonic if and only if its tension vanishes: $tr(\nabla d\phi) = 0$. If M is Riemann surface then ϕ is harmonic if and only if on every local complex coordinate z on M the following equation holds

$$\nabla^{\phi}_{\frac{\partial}{\partial \bar{z}}} d\phi\left(\frac{\partial}{\partial z}\right) = 0, \tag{26}$$

where the left hand of this equation is just a non-zero multiple of the tension field of ϕ.

For our purposes we need a reformulation of Eq. (26) in terms of the Maurer–Cartan form or Moment map β of $\mathcal{Z}$, see [5,7].

Let D be the *canonical connection of second kind*, i.e. the affine connection on $\mathcal{Z}$ for which the parallel transport along the curve $t \to \exp(tX).x$ is realized by $d\exp(tX)$. Hence at the basepoint $o \in \mathcal{Z}$ we have

$$D_{X^*}Y^*(o) = \frac{d}{dt}\bigg|_{t=0} d\exp(-tX)Y^* = \left[X^*, Y^*\right](o) = -[X, Y]_{\mathfrak{p}}. \tag{27}$$

Note also that D is determined by the condition $(D_{X^*}X^*)_o = 0,\ \forall X \in \mathfrak{p}$. Since $\nabla \neq D$ and $D\langle\, ,\rangle = 0$, D has non-vanishing torsion. From (27) we obtain

$$T^D_o\left(X^*, Y^*\right) = -[X, Y]_{\mathfrak{p}}, \quad X, Y \in \mathfrak{p}. \tag{28}$$

The following formula allows to compute D in terms of β and the Lie algebra structure of $\mathfrak{so}(3,1)$.

Lemma 4.1. *(See [7].)*

$$\beta(D_X Y) = X\beta(Y) - \left[\beta(X), \beta(Y)\right], \quad X, Y \in \mathcal{X}(\mathcal{Z}). \tag{29}$$

Let us now compute the Levi-Civita connection ∇ of the normal metric on $\mathcal{Z}$. Since $P : SO_o(3,1) \to \mathcal{Z}$ is a (pseudo) Riemannian submersion, P sends $\mathfrak{p}$-horizontal geodesics in $SO_o(3,1)$ onto ∇-geodesics in $\mathcal{Z}$. Hence $\nabla_{X^*}X^* = 0$ for every $X \in \mathfrak{p}$ which implies $\nabla_{X^*}Y^* + \nabla_{Y^*}X^* = 0$, for any $X, Y \in \mathfrak{p}$. Now since ∇ is torsionless we get

$$\nabla_{X^*}Y^* = \frac{1}{2}\left[X^*, Y^*\right], \quad \forall X, Y \in \mathfrak{p}. \tag{30}$$

Hence at $o \in \mathcal{Z}$ we get

$$\left(\nabla_{X^*}Y^*\right)(o) = -\frac{1}{2}[X, Y]_{\mathfrak{p}}, \quad \forall X, Y \in \mathfrak{p}. \tag{31}$$

We are now ready to obtain a formula for the Levi-Civita connection ∇ on $\mathcal{Z}$ in terms of β namely,

Lemma 4.2.

$$\beta(\nabla_X Y) = X\beta(Y) - \left[\beta(X), \beta(Y)\right] + \frac{1}{2}\pi_{\mathfrak{p}}\left(\left[\beta(X), \beta(Y)\right]\right), \quad X, Y \in \mathcal{X}(\mathcal{Z}), \tag{32}$$

where $\pi_{\mathfrak{p}} : \mathcal{Z} \times \mathfrak{so}(3,1) \to SO_o(3,1) \times_K \mathfrak{p} \equiv T\mathcal{Z}$ is the projection onto the tangent bundle of $\mathcal{Z}$.

Proof. Let X^*, Y^* be Killing vector fields on $\mathcal{Z}$ determined by $X, Y \in \mathfrak{p}$.

From de definition of β, (27) and (31) we have

$$\beta\left(\left(\nabla_{X^*}Y^*\right)(o)\right) - \beta\left(\left(D_{X^*}Y^*\right)(o)\right) = -\frac{1}{2}[X, Y]_{\mathfrak{p}} + [X, Y]_{\mathfrak{p}} = \frac{1}{2}[X, Y]_{\mathfrak{p}}, \quad \forall X, Y \in \mathfrak{p}.$$

On the other hand the difference tensor $\nabla - D$ is $SO_o(3,1)$-invariant, and so is $\beta(\nabla - D) = \beta(\nabla) - \beta(D)$ by formula (18). Hence it is determined by its value at the point $o \in \mathcal{Z}$. Thus formula (32) follows. □

Define the D-fundamental form of $\phi : M \to \mathcal{Z}$ by

$$D\,d\phi(U, V) = D^{\phi}_U d\phi(V) - d\phi\left(\nabla^M_U V\right), \quad U, V \in \mathcal{X}(M), \tag{33}$$

in which D^{ϕ} is the connection on $\phi^{-1}T\mathcal{Z}$ determined by D and ∇^M is the Levi-Civita connection on M determined by a conformal metric. The map $\phi : M \to \mathcal{Z}$ is called D-harmonic if and only if $tr(D\,d\phi) = 0$ or equivalently

$$D^{\phi}_{\frac{\partial}{\partial \bar{z}}} d\phi\left(\frac{\partial}{\partial z}\right) = 0.$$

Lemma 4.3. *$\phi : M \to \mathcal{Z}$ is harmonic if and only if*

$$\bar\partial(\phi^*\beta)' - \left[(\phi^*\beta)'' \wedge (\phi^*\beta)'\right] = 0, \tag{34}$$

where $\phi^*\beta = (\phi^*\beta)' + (\phi^*\beta)''$ is the decomposition of the (complex) one form $\phi^*\beta$ into one forms of type $(1,0)$ and $(0,1)$.

Proof. It is consequence of the following formula for the tension of ϕ

$$-\beta\left(\nabla_{\frac{\partial}{\partial\bar z}} d\phi\left(\frac{\partial}{\partial z}\right)\right) dz \wedge d\bar z = \bar\partial(\phi^*\beta)' - \left[(\phi^*\beta)'' \wedge (\phi^*\beta)'\right]. \tag{35}$$

To obtain formula (35) we note first that from (32) we have $\beta(tr\,\nabla d\phi) = \beta(tr\,D\,d\phi)$ which since M is a Riemann surface is equivalent to

$$\beta\left(\nabla^{\phi}_{\frac{\partial}{\partial\bar z}} d\phi\left(\frac{\partial}{\partial z}\right)\right) = \beta\left(D^{\phi}_{\frac{\partial}{\partial\bar z}} d\phi\left(\frac{\partial}{\partial z}\right)\right).$$

On the other hand from formula (29) we obtain

$$\beta\left(D^{\phi}_{\frac{\partial}{\partial\bar z}} d\phi\left(\frac{\partial}{\partial z}\right)\right) = \frac{\partial}{\partial\bar z}\beta\,d\phi\left(\frac{\partial}{\partial z}\right) - \left[\beta\,d\phi\left(\frac{\partial}{\partial\bar z}\right), \beta\,d\phi\left(\frac{\partial}{\partial z}\right)\right], \tag{36}$$

from which (35) follows. □

5. Harmonic twistor lifts

A natural question is to characterize those conformaly immersed surfaces $f : M \to \mathbb{S}^3_1$ whose twistor lift $\hat f : M \to \mathcal{Z}$ is a harmonic map.

Let $f : M \to \mathbb{S}^3_1$ be a conformal immersed surface and $F \in SO_o(3,1)$ be a frame of f which is adapted to the surface f, i.e. F is a locally defined map on an open subset $U \subset M$ satisfying

$$f(x) = F_1(x), \qquad F_4(x) = \mathbf{n}(x),$$
$$\operatorname{span}\{F_2(x), F_3(x)\} = df_x(T_xM), \quad \forall x \in U,$$

where $F_i = F.e_i$ are the columns of the matrix F. From the structure equations (4) of the immersed spacelike surface f we obtain the following evolution equations of the frame F

$$\begin{aligned}
f_z &= \frac{e^u}{\sqrt2}(F_2 - iF_3),\\
(F_2)_z &= n - \frac{e^u}{\sqrt2}.f - iu_zF_3 + \left(\frac{e^{-u}\xi + e^uH}{\sqrt2}\right).\mathbf{n},\\
(F_3)_z &= i\frac{e^u}{\sqrt2}.f + iu_zF_2 + i\left(\frac{e^{-u}\xi - e^uH}{\sqrt2}\right).\mathbf{n},\\
\mathbf{n}_z &= \left(\frac{e^{-u}\xi + e^uH}{\sqrt2}\right).F_2 + i\left(\frac{e^{-u}\xi - e^uH}{\sqrt2}\right).F_3.
\end{aligned} \tag{37}$$

Written in matrix form these equations take the form

$$F_z = FA, \qquad F_{\bar z} = FB, \tag{38}$$

where the complex matrices $A, B = \bar A$ are given by

$$A = \begin{pmatrix} 0 & -\frac{e^u}{\sqrt2} & i\frac{e^u}{\sqrt2} & 0\\ \frac{e^u}{\sqrt2} & 0 & iu_z & \frac{e^{-u}\xi+e^uH}{\sqrt2}\\ -i\frac{e^u}{\sqrt2} & -iu_z & 0 & i(\frac{e^{-u}\xi-e^uH}{\sqrt2})\\ 0 & \frac{e^{-u}\xi+e^uH}{\sqrt2} & i(\frac{e^{-u}\xi-e^uH}{\sqrt2}) & 0 \end{pmatrix}, \tag{39}$$

$$B = \begin{pmatrix} 0 & -\frac{e^u}{\sqrt2} & -i\frac{e^u}{\sqrt2} & 0\\ \frac{e^u}{\sqrt2} & 0 & -iu_{\bar z} & \frac{e^{-u}\bar\xi+e^uH}{\sqrt2}\\ i\frac{e^u}{\sqrt2} & iu_{\bar z} & 0 & -i(\frac{e^{-u}\bar\xi-e^uH}{\sqrt2})\\ 0 & \frac{e^{-u}\bar\xi+e^uH}{\sqrt2} & -i(\frac{e^{-u}\bar\xi-e^uH}{\sqrt2}) & 0 \end{pmatrix}. \tag{40}$$

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The pull-back $\alpha = F^{-1}\,dF$ of the Maurer–Cartan form of $SO_o(3,1)$ is given in terms of A, B by

$$\alpha = A\,dz + B\,d\bar{z}. \tag{41}$$

Taking $\mathfrak{p}$ and $\mathfrak{k}$ projections we obtain $\alpha = \alpha_{\mathfrak{k}} + \alpha_{\mathfrak{p}}$, with

$$\alpha_{\mathfrak{k}} = A_{\mathfrak{k}}\,dz + B_{\mathfrak{k}}\,d\bar{z}, \qquad \alpha_{\mathfrak{p}} = A_{\mathfrak{p}}\,dz + B_{\mathfrak{p}}\,d\bar{z}, \tag{42}$$

in which

$$A_{\mathfrak{p}} = \begin{pmatrix} 0 & -\frac{e^u}{\sqrt{2}} & i\frac{e^u}{\sqrt{2}} & 0 \\ \frac{e^u}{\sqrt{2}} & 0 & 0 & \frac{e^{-u}\xi + e^u H}{\sqrt{2}} \\ -i\frac{e^u}{\sqrt{2}} & 0 & 0 & i(\frac{e^{-u}\xi - e^u H}{\sqrt{2}}) \\ 0 & \frac{e^{-u}\xi + e^u H}{\sqrt{2}} & i(\frac{e^{-u}\xi - e^u H}{\sqrt{2}}) & 0 \end{pmatrix}, \tag{43}$$

$$B_{\mathfrak{p}} = \begin{pmatrix} 0 & -\frac{e^u}{\sqrt{2}} & -i\frac{e^u}{\sqrt{2}} & 0 \\ \frac{e^u}{\sqrt{2}} & 0 & 0 & \frac{e^{-u}\bar{\xi} + e^u H}{\sqrt{2}} \\ i\frac{e^u}{\sqrt{2}} & 0 & 0 & -i(\frac{e^{-u}\bar{\xi} - e^u H}{\sqrt{2}}) \\ 0 & \frac{e^{-u}\bar{\xi} + e^u H}{\sqrt{2}} & -i(\frac{e^{-u}\bar{\xi} - e^u H}{\sqrt{2}}) & 0 \end{pmatrix}, \tag{44}$$

$$A_{\mathfrak{k}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & iu_z & 0 \\ 0 & -iu_z & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad B_{\mathfrak{k}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -iu_{\bar{z}} & 0 \\ 0 & iu_{\bar{z}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{45}$$

Also since M is a Riemann surface we decompose $\alpha_{\mathfrak{p}}$ and $\alpha_{\mathfrak{k}}$ into its $(1,0)$ and $(0,1)$ parts

$$\alpha_{\mathfrak{p}} = \alpha'_{\mathfrak{p}} + \alpha''_{\mathfrak{p}}, \qquad \alpha_{\mathfrak{k}} = \alpha'_{\mathfrak{k}} + \alpha''_{\mathfrak{k}}.$$

In terms of the above matrices we have

$$\alpha'_{\mathfrak{k}} = A_{\mathfrak{k}}\,dz, \qquad \alpha''_{\mathfrak{k}} = B_{\mathfrak{k}}\,d\bar{z}, \qquad \alpha'_{\mathfrak{p}} = A_{\mathfrak{p}}\,dz, \qquad \alpha''_{\mathfrak{p}} = B_{\mathfrak{p}}\,d\bar{z}. \tag{46}$$

Note that the adapted frame F of f is also a frame of the twistor lift $\hat{f}$ since $F.o = (F_1, F_4) = (f, \mathbf{n}) = \hat{f}$, where $o = (e_1, e_4) \in \mathcal{Z}$ is the fixed basepoint. From formula $\hat{f}^*\beta = Ad(F)\alpha_{\mathfrak{p}}$ we obtain

$$\big(\hat{f}^*\beta\big)' = Ad(F)\alpha'_{\mathfrak{p}}, \qquad \big(\hat{f}^*\beta\big)'' = Ad(F)\alpha''_{\mathfrak{p}}.$$

Now using the identity (see [4, p. 241])

$$d\,Ad(F) = Ad(F) \circ ad\,\alpha,$$

we compute

$$\bar{\partial}\big(\hat{f}^*\beta\big)' = \bar{\partial}\big\{Ad(F)\alpha'_{\mathfrak{p}}\big\} = Ad(F)\big\{\bar{\partial}\alpha'_{\mathfrak{p}} + \big[\alpha_{\mathfrak{k}} \wedge \alpha'_{\mathfrak{p}}\big] + \big[\alpha''_{\mathfrak{p}} \wedge \alpha'_{\mathfrak{p}}\big]\big\}. \tag{47}$$

On the other hand

$$\big[\big(\hat{f}^*\beta\big)'' \wedge (\hat{f}^*\beta)'\big] = Ad(F)\big[\alpha''_{\mathfrak{p}} \wedge \alpha'_{\mathfrak{p}}\big]. \tag{48}$$

Therefore

$$\bar{\partial}\big(\hat{f}^*\beta\big)' - \big[\big(\hat{f}^*\beta\big)'' \wedge \big(\hat{f}^*\beta\big)'\big] = Ad(F)\big\{\bar{\partial}\alpha'_{\mathfrak{p}} + \big[\alpha_{\mathfrak{k}} \wedge \alpha'_{\mathfrak{p}}\big]\big\}.$$

Thus a direct consequence of Lemma 5 is

Proposition 5.1. *Let $f : M \to \mathbb{S}^3_1$ be a conformal immersed surface and let $\hat{f} : M \to \mathcal{Z}$ be its twistor lift. Then $\hat{f}$ is a harmonic map if and only if for every adapted frame F of f the $\mathfrak{so}(3,1)$-valued one form $\alpha = F^{-1}\,dF$ satisfies*

$$\bar{\partial}\alpha'_{\mathfrak{p}} + \big[\alpha_{\mathfrak{k}} \wedge \alpha'_{\mathfrak{p}}\big] = 0. \tag{49}$$

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A reformulation of (49) in terms of the evolution matrices A, B is easily obtained. Recall that $\alpha'_{\mathfrak p} = A_{\mathfrak p}\,dz$ and $\alpha''_{\mathfrak p} = B_{\mathfrak p}\,d\bar z$, hence

$$\bar\partial\alpha'_{\mathfrak p} + [\alpha_{\mathfrak k} \wedge \alpha'_{\mathfrak p}] = -\left(\frac{\partial}{\partial\bar z}A_{\mathfrak p} + [B_{\mathfrak k}, A_{\mathfrak p}]\right) dz \wedge d\bar z. \tag{50}$$

Corollary 5.1. *Let $f : M \to \mathbb{S}^3_1$ be a conformal immersion and let $\hat f : M \to \mathcal{Z}$ be the twistor lift of f. Then $\hat f$ is harmonic if and only if for every adapted frame F of f the complex matrices A, B defined by $\alpha = F^{-1}\,dF = A\,dz + B\,d\bar z$ satisfy*

$$\frac{\partial}{\partial\bar z}A_{\mathfrak p} + [B_{\mathfrak k}, A_{\mathfrak p}] = 0. \tag{51}$$

From the explicit form of the matrices $A_{\mathfrak p}$ and $B_{\mathfrak k}$ in (43) and (45) we conclude from Eq. (51) that $\hat f$ is harmonic if and only if

$$\begin{aligned}&\frac{\partial}{\partial\bar z}\left(e^{-u}\xi + e^{u}H\right) + u_{\bar z}\left(e^{-u}\xi - e^{u}H\right) = 0,\\ &\frac{\partial}{\partial\bar z}\left(e^{-u}\xi - e^{u}H\right) + u_{\bar z}\left(e^{-u}\xi + e^{u}H\right) = 0.\end{aligned} \tag{52}$$

Cancelling terms we are left with

$$\begin{aligned}&e^{-u}\xi_{\bar z} + e^{u}H_{\bar z} = 0,\\ &e^{-u}\xi_{\bar z} - e^{u}H_{\bar z} = 0.\end{aligned} \tag{53}$$

Combining with Codazzi's equation $\xi_{\bar z} = e^{2u}H_z$, system (53) is equivalent to

$$\begin{aligned}&e^{u}(H_z + H_{\bar z}) = 0,\\ &e^{u}(H_z - H_{\bar z}) = 0,\end{aligned} \tag{54}$$

hence $H_x = H_y = 0$ and conversely. We have thus proved the following

Theorem 5.2. *Let $f : M \to \mathbb{S}^3_1$ be a conformal (hence spacelike) immersion and let $\hat f : M \to \mathcal{Z}$ be its twistor lift. Let $\langle\, .\, ,\, .\,\rangle$ be the normal metric on the twistor space $\mathcal{Z}$. Then $\hat f : M \to (\mathcal{Z}, \langle\, .\, ,\, .\,\rangle)$ is a harmonic map if and only if the immersed surface f has constant mean curvature.*

By Codazzi's equation $\hat f$ is harmonic if and only if the Hopf complex differential $q = \xi\,dz^2 = -\langle f_{zz}, \mathbf{n}\rangle^c\,dz \otimes dz$ is holomorphic.

6. One parameter deformations

Here we show that harmonic twistor lifts exist within a family parameterized by the complex numbers of unit modulus.

Let $f : M \to \mathbb{S}^3_1$ a conformal immersion and F a (local) adapted frame of f, hence F is also a frame of $\hat f$. Let $\alpha = F^{-1}\,dF$ be the pullback of the Maurer–Cartan form by F. According to the reductive decomposition $\mathfrak{so}(3,1) = \mathfrak k \oplus \mathfrak p$ we decompose as before $\alpha = \alpha_{\mathfrak k} + \alpha_{\mathfrak p}$ where in terms of matrices $A_{\mathfrak p}, B_{\mathfrak p}, A_{\mathfrak k}, B_{\mathfrak k}$ (43), (44) and (45) these forms are expressed by

$$\alpha_{\mathfrak k} = A_{\mathfrak k}\,dz + B_{\mathfrak k}d\bar z, \qquad \alpha_{\mathfrak p} = A_{\mathfrak p}\,dz + B_{\mathfrak p}d\bar z, \qquad \alpha'_{\mathfrak p} = A_{\mathfrak p}\,dz, \qquad \alpha''_{\mathfrak p} = B_{\mathfrak p}\,d\bar z.$$

On the other hand α satisfies the Maurer–Cartan equation $d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0$ which splits up into

$$\begin{aligned}&\bar\partial\alpha'_{\mathfrak p} + [\alpha_{\mathfrak k} \wedge \alpha'_{\mathfrak p}] + \partial\alpha''_{\mathfrak p} + [\alpha_{\mathfrak k} \wedge \alpha''_{\mathfrak p}] + [\alpha'_{\mathfrak p} \wedge \alpha''_{\mathfrak p}]_{\mathfrak p} = 0,\\ &d\alpha_{\mathfrak k} + \frac{1}{2}[\alpha_{\mathfrak k} \wedge \alpha_{\mathfrak k}] + [\alpha'_{\mathfrak p} \wedge \alpha''_{\mathfrak p}]_{\mathfrak k} = 0.\end{aligned} \tag{55}$$

We give below a further property of the geometry of our twistor lifts $\hat f$ which is consequence of the form of the structure equations of the immersion f which is reflected in the matrices $A_{\mathfrak p}$ (43) and $\overline{A}_{\mathfrak p} = B_{\mathfrak p}$ (44).

Lemma 6.1. *Let $f : M \to \mathbb{S}^3_1$ be a conformal immersion, F an arbitrary adapted frame of f, and $z = x + iy$ a local complex coordinate on M. Set $F^{-1}F_z = A_{\mathfrak p} + A_{\mathfrak k}$ and $F^{-1}F_{\bar z} = B_{\mathfrak p} + B_{\mathfrak k}$, where the complex matrices $A_{\mathfrak p}$, $B_{\mathfrak p}$ given by (43), (44). Then the one forms $\alpha'_{\mathfrak p} = A_{\mathfrak p}\,dz$ and $\alpha''_{\mathfrak p} = B_{\mathfrak p}\,d\bar z$ satisfy*

$$[\alpha'_{\mathfrak p} \wedge \alpha''_{\mathfrak p}]_{\mathfrak p} = 0. \tag{56}$$

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Proof. From the structure equations (37) of the immersion f we obtain $\langle f_z, \mathbf{n}_{\bar z}\rangle^c = \langle f_{\bar z}, \mathbf{n}_z\rangle^c$ and $\langle \frac{\partial}{\partial z} F_i - [\frac{\partial}{\partial z} F_i]^T, \mathbf{n}_{\bar z}\rangle^c = 0$, $i = 2, 3$, where $[\frac{\partial}{\partial z} F_i]^T$ denotes the projection onto the tangent bundle of the immersed surface. These equations clearly imply $[A_{\mathfrak p}, B_{\mathfrak p}]_{\mathfrak p} = 0$. Hence

$$[\alpha'_{\mathfrak p} \wedge \alpha''_{\mathfrak p}]_{\mathfrak p} = [A_{\mathfrak p}, B_{\mathfrak p}]_{\mathfrak p}\, dz \wedge d\bar z = 0. \qquad \square$$

Assume now that $f : M \to \mathbb{S}^3_1$ has constant mean curvature. Thus $\hat f$ is a harmonic map by Theorem 5.2, and so it satisfies the harmonic map equation

$$0 = \bar\partial \alpha'_{\mathfrak p} + [\alpha_{\mathfrak k} \wedge \alpha'_{\mathfrak p}] = -\left(\frac{\partial}{\partial \bar z} A_{\mathfrak p} + [B_{\mathfrak k}, A_{\mathfrak p}]\right) dz \wedge d\bar z.$$

Taking into account condition (56) the first equation in (55) reduces to

$$\partial \alpha''_{\mathfrak p} + [\alpha_{\mathfrak k} \wedge \alpha''_{\mathfrak p}] = 0.$$

Hence the pair of Eqs. (55) become

$$(\mathbf{a}) \quad \partial \alpha''_{\mathfrak p} + [\alpha_{\mathfrak k} \wedge \alpha''_{\mathfrak p}] = 0,$$
$$(\mathbf{b}) \quad d\alpha_{\mathfrak k} + \frac{1}{2}[\alpha_{\mathfrak k} \wedge \alpha_{\mathfrak k}] + [\alpha'_{\mathfrak p} \wedge \alpha''_{\mathfrak p}] = 0.$$

For $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ set

$$\lambda.\alpha = \alpha_\lambda = \lambda^{-1}\alpha'_{\mathfrak p} + \alpha_{\mathfrak k} + \lambda \alpha''_{\mathfrak p}. \tag{57}$$

Due to $\overline{A}_{\mathfrak p} = B_{\mathfrak p}$ and $\overline{A}_{\mathfrak k} = B_{\mathfrak k}$, α_λ is $\mathfrak{so}(3,1)$-valued for every $\lambda \in \mathbb{S}^1$. Moreover $\lambda.\alpha = \alpha_\lambda$ defines an action of $\mathbb{S}^1$ on $\mathfrak{so}(3,1)$-valued 1-forms which leaves invariant the solution set of Eqs. (**a**) and (**b**) above. Comparing coefficients of λ it follows that Eqs. (**a**) and (**b**) above hold for α if and only if α_λ satisfies

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0, \quad \forall \lambda \in \mathbb{S}^1.$$

This is the so-called *zero curvature condition* (ZCC) [4]. In this way the harmonic map equation for twistor lifts to $\mathcal{Z}$ is encoded in a loop of "zero curvature" equations.

Now let us assume that the Riemann surface M is simply connected (otherwise we pass to its universal covering space $\tilde M$), and fix a base point $m_o \in M$. Then for each $\lambda \in \mathbb{S}^1$ we can integrate and solve

$$dF_\lambda = F_\lambda \alpha_\lambda, \qquad F_\lambda(m_o) = Id. \tag{58}$$

The solution map $F_\lambda = (f_\lambda, (F_\lambda)_2, (F_\lambda)_3, \mathbf{n}_\lambda) : M \to SO_o(3,1)$ is called an *extended frame* and satisfies

$$F_\lambda^{-1}(F_\lambda)_z = \lambda^{-1}A_{\mathfrak p} + A_{\mathfrak k}, \qquad F_\lambda^{-1}(F_\lambda)_{\bar z} = \lambda B_{\mathfrak p} + B_{\mathfrak k}, \quad \forall \lambda \in \mathbb{S}^1. \tag{59}$$

Moreover since $(\alpha_\lambda)_{\mathfrak p} = \lambda^{-1}\alpha'_{\mathfrak p} + \lambda\alpha''_{\mathfrak p} = (\alpha_\lambda)'_{\mathfrak p} + (\alpha_\lambda)''_{\mathfrak p}$, and $(\alpha_\lambda)_{\mathfrak k} = \alpha_{\mathfrak k}$, then the one form α_λ satisfies Eqs. (**a**) and (**b**) for every $\lambda \in \mathbb{S}^1$. Thus if $P : SO_o(3,1) \to \mathcal{Z}$ denotes the projection map $P(g) = g.o$, then $\phi_\lambda = P \circ F_\lambda : M \to \mathcal{Z}$ is harmonic $\forall \lambda \in \mathbb{S}^1$.

The family $\{\phi_\lambda, \lambda \in \mathbb{S}^1\}$ is called *the associated family* of the harmonic map $\hat f$ [7]. Note that $\phi_{\{\lambda=1\}} = \hat f$, hence each ϕ_λ is a deformation of $\hat f$.

Let $f_\lambda = \pi \circ \phi_\lambda : M \to \mathbb{S}^3_1$, hence from (59) we extract

$$(f_\lambda)_z = \lambda^{-1}\frac{e^u}{\sqrt 2}[(F_\lambda)_2 - i(F_\lambda)_3],$$
$$(\mathbf{n}_\lambda)_z = \lambda^{-1}\left[\left(\frac{e^{-u}\xi + e^u H}{\sqrt 2}\right)(F_\lambda)_2 + i\left(\frac{e^{-u}\xi - e^u H}{\sqrt 2}\right)(F_\lambda)_3\right]. \tag{60}$$

From the first equation above we get

$$\langle (f_\lambda)_z, (f_\lambda)_z\rangle = \langle (f_\lambda)_z, (f_\lambda)_{\bar z}\rangle^c = e^{2u},$$

thus $\{f_\lambda, \lambda \in \mathbb{S}^1\}$ is a family of conformal immersions into $\mathbb{S}^3_1$, with a common conformal factor u, hence all f_λ induce the same metric for every $\lambda \in \mathbb{S}^1$. Let H_λ be the mean curvature of f_λ. Since u is the conformal parameter of f_λ, we get from (59),

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$$H_\lambda = -e^{-2u}\langle (f_\lambda)_{\bar z z}, \mathbf{n}_\lambda\rangle = e^{-2u}\langle (f_\lambda)_z, (\mathbf{n}_\lambda)_z\rangle = H, \quad \forall \lambda \in \mathbb{S}^1.$$

Thus H_λ does not depend on λ. On the other hand let $q_\lambda = \xi_\lambda\, dz \otimes dz$ be the Hopf complex differential of f_λ. Then from (59) and (60) we get

$$\xi_\lambda = -\langle (f_\lambda)_{zz}, \mathbf{n}_\lambda\rangle^c = \langle (f_\lambda)_z, (\mathbf{n}_\lambda)_z\rangle^c = \lambda^{-2}\xi. \tag{61}$$

Gauss equation for f_λ reads

$$K_\lambda = 1 - H^2 + |\xi_\lambda|^2 e^{-4u} = 1 - H^2 + |\xi|^2 e^{-4u} = K. \tag{62}$$

Hence all $f_\lambda : M \to \mathbb{S}^3_1$ are isometric surfaces. Summing up we have proved the following

Theorem 6.1. *Let $f : M \to \mathbb{S}^3_1$ be a conformal immersion with constant mean curvature H, Gaussian curvature K and Hopf complex differential $q = \xi\, dz \otimes dz$. Let $\hat f : M \to \mathcal{Z}$ be its twistor lift. Then there is a one parameter family of harmonic maps $\phi_\lambda : M \to \mathcal{Z}$, $\lambda \in \mathbb{S}^1$ satisfying $\phi_{\{\lambda=1\}} = \hat f$, which are given by $\phi_\lambda = F_\lambda.o$, where the extended frame $F_\lambda : M \to SO_o(3,1)$ solves (58).*

The projection $f_\lambda = \pi \circ \phi_\lambda : M \to \mathbb{S}^3_1$ is an $\mathbb{S}^1$-family of isometric conformally immersed surfaces satisfying $f_{\{\lambda=1\}} = f$ and $\hat f_\lambda = \phi_\lambda$, $\forall \lambda \in \mathbb{S}^1$. Moreover, the induced metric $f^*_\lambda\langle .,. \rangle$ does not depend on λ and all f_λ have constant mean curvature H, Gaussian curvature K and Hopf complex differential $q_\lambda = \lambda^{-2}\xi\, dz \otimes dz$.

Remark 6.1. $\mathcal{Z}$ is a reductive homogeneous space which is not symmetric since $[\mathfrak{p},\mathfrak{p}] \not\subset \mathfrak{k}$. Thus the harmonic map equation for arbitrary maps into $\mathcal{Z}$ cannot be encoded in a loop of connections with zero curvature. Nevertheless since twistor lifts satisfy condition (56) the harmonic map equation for twistor lifts admits a formulation as the flatness condition (ZCC) of a family of connections parameterized by unit complex numbers. This reflects the complete integrability of the harmonic map equation for twistor lifts.

6.1. Holomorphic twistor lifts

Here we consider the behaviour of twistor lifts in relation to both invariant almost complex structures J', J'' on the horizontal distribution $\mathfrak{h} \subset \mathcal{Z}$ introduced before. A smooth map $\phi : M \to \mathcal{Z}$ is said *horizontal* if $d\phi(T_x M) \subset \mathfrak{h}_{\phi(x)}$ for any $x \in M$. From the structure of matrices (43) and (44) we conclude that twistor lifts are horizontal maps. Let J be one of the almost complex structures J', J'' considered before. A horizontal map $\phi : M \to \mathcal{Z}$ is J-holomorphic if it satisfies a Cauchy–Riemann type equation

$$J \circ d\phi = d\phi \circ J^M, \tag{63}$$

where J^M is the complex structure of M. Equivalently ϕ is J-holomorphic if and only if $d\phi(T^{(1,0)}M) \subset \mathfrak{h}^{(1,0)}_\phi$, where

$$\mathfrak{h}^{(1,0)}_q = \{X \in \mathfrak{h}^{\mathbb{C}}_q :\ JX = iX\}.$$

Recall now that the isomorphism $\beta : T\mathcal{Z} \to [\mathfrak{p}]$ constructed before satisfies (20), i.e.

$$\beta_{g.o}(\mathfrak{h}_{g.o}) = \{g.o\} \times Ad(g)\mathcal{H}, \quad \forall g \in SO_o(3,1).$$

Then a horizontal map $\phi : M \to \mathcal{Z}$ is J-holomorphic if and only if for every frame F of ϕ

$$\phi^*\beta\left(\frac{\partial}{\partial z}\right) \in Ad(F)\mathcal{H}^{(1,0)}.$$

On the other hand for every frame F of ϕ the following identity holds

$$\phi^*\beta\left(\frac{\partial}{\partial z}\right) = Ad(F)\alpha_{\mathfrak{p}}\left(\frac{\partial}{\partial z}\right),$$

in which $\alpha_{\mathfrak{p}}$ is the $\mathfrak{p}$-component of the Maurer–Cartan one form $\alpha = F^{-1}dF$. We conclude that a horizontal map $\phi : M \to \mathcal{Z}$ is J-holomorphic if and only if for every frame F of ϕ

$$\alpha_{\mathfrak{p}}\left(\frac{\partial}{\partial z}\right) \in \mathcal{H}^{(1,0)}.$$

From the explicit form of J' at $o \in \mathcal{Z}$ we see that the i-eigenspace $\mathcal{H}^{(1,0)}$ corresponding to J' consists of matrices of the form

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$$\begin{pmatrix} 0 & a & -ia & 0 \\ -a & 0 & 0 & b \\ ia & 0 & 0 & -ib \\ 0 & b & -ib & 0 \end{pmatrix}, \quad a, b \in \mathbb{C}.$$

Now let $f : M \to \mathbb{S}^3_1$ be a conformal (hence spacelike) immersed surface and $\hat{f} : M \to \mathcal{Z}$ its twistor lift. Thus $\hat{f}$ is J-holomorphic if and only if $\alpha_\mathfrak{p}(\frac{\partial}{\partial z}) \in \mathcal{H}^{(1,0)}$ for every adapted frame F of f. From (43) we have

$$\alpha_\mathfrak{p}\left(\frac{\partial}{\partial z}\right) = A_\mathfrak{p} = \begin{pmatrix} 0 & -\frac{e^u}{\sqrt{2}} & i\frac{e^u}{\sqrt{2}} & 0 \\ \frac{e^u}{\sqrt{2}} & 0 & 0 & \frac{e^{-u}\xi + e^u H}{\sqrt{2}} \\ -i\frac{e^u}{\sqrt{2}} & 0 & 0 & i(\frac{e^{-u}\xi - e^u H}{\sqrt{2}}) \\ 0 & \frac{e^{-u}\xi + e^u H}{\sqrt{2}} & i(\frac{e^{-u}\xi - e^u H}{\sqrt{2}}) & 0 \end{pmatrix},$$

which is in the i-eigenspace $\mathcal{H}^{(1,0)}$ of J' if and only if $i(e^{-u}\xi - e^u H) = -i(e^{-u}\xi + e^u H)$, or $2e^{-u}\xi = 0$, hence $\xi = 0$.

Arguing in an analogous way we conclude that a horizontal map $\phi : M \to \mathcal{Z}$ is J''-holomorphic if and only if for every frame F of ϕ the complex matrix $\alpha_\mathfrak{p}(\frac{\partial}{\partial z}) = A_\mathfrak{p}$ is an eigenvector of J'' corresponding to the eigenvalue i. Using the explicit form of J'' (24) we see that $X \in \mathcal{H}^\mathbb{C}$ satisfies $J''X = iX$ if and only if

$$X = \begin{pmatrix} 0 & a & -ia & 0 \\ -a & 0 & 0 & b \\ ia & 0 & 0 & ib \\ 0 & b & ib & 0 \end{pmatrix}, \quad a, b \in \mathbb{C}.$$

Turning to the particular case of the twistor lift $\hat{f}$ of an immersed surface f, we see that $\hat{f}$ is J''-holomorphic if and only if $J''A_\mathfrak{p} = iA_\mathfrak{p}$ if and only if $i(e^{-u}\xi - e^u H) = i(e^{-u}\xi + e^u H)$, or $2ie^u H = 0$.

One can also characterize twistor lifts which are *conformal* maps. For, let $f : M \to \mathbb{S}^3_1$ be a conformal immersion and take an adapted (local) frame F of f. Thus $\hat{f}^*\beta(\frac{\partial}{\partial z}) = Ad(F)A_\mathfrak{p}$, where the complex matrix $A_\mathfrak{p}$ is given by (43). Since β preserves the metric, we compute

$$\left\langle \hat{f}_*\left(\frac{\partial}{\partial z}\right), \hat{f}_*\left(\frac{\partial}{\partial z}\right)\right\rangle^c = \left\langle \hat{f}^*\beta\left(\frac{\partial}{\partial z}\right), \hat{f}^*\beta\left(\frac{\partial}{\partial z}\right)\right\rangle^c = \langle Ad(F)A_\mathfrak{p}, Ad(F)A_\mathfrak{p}\rangle^c = \langle A_\mathfrak{p}, A_\mathfrak{p}\rangle^c = -\frac{1}{2}tr(A_\mathfrak{p}^2) = -2\xi H.$$

Thus $\hat{f}$ is conformal if and only if it is J' or J'' holomorphic. We have thus obtained the following.

Proposition 6.1. *Let $f : M \to \mathbb{S}^3_1$ be an immersed spacelike surface and let $\hat{f} : M \to \mathcal{Z}$ its twistor lift. Then*

i) *$\hat{f}$ is J'-holomorphic if and only if f is totally umbilic ($\xi \equiv 0$).*
ii) *$\hat{f}$ is J''-holomorphic if and only if f has vanishing mean curvature ($H \equiv 0$).*
iii) *$\hat{f}$ is conformal if and only if f satisfies $\xi.H = 0$. In particular if $\hat{f}$ is conformal, then it is harmonic.*

Remark 6.2. As consequence of Theorem 5.2 and Codazzi's equation, it follows that J' and J''-holomorphic twistor lifts $\hat{f}$ are harmonic maps. Moreover from the proof of Theorem 6.1 it follows that the one parameter deformation introduced preserves J'- and J''-holomorphicity.

7. On the twistor energy

In order to gain some insight of the energy (25) we compute the twistor energy and study its behaviour for (compact) genus zero and genus one spacelike surfaces.

Let $f : M \to \mathbb{S}^3_1$ be a conformal immersion, hence the energy density $\|d\hat{f}\|^2$ of the twistor lift is by definition

$$\|d\hat{f}\|^2(p) = \langle d\hat{f}(e_1), d\hat{f}(e_1)\rangle + \langle d\hat{f}(e_2), d\hat{f}(e_2)\rangle,$$

where $\{e_1, e_2\}$ is any orthonormal basis of T_pM. To compute $\|d\hat{f}\|^2$ we use the induced metric $g = f^*\langle\ ,\ \rangle$ which is conformal and locally given by $g = 2e^{2u}\, dz \otimes d\bar{z}$, where u is the conformal factor. Thus

$$\frac{1}{2}\|d\hat{f}\|^2 = e^{-2u}\langle \hat{f}_z, \hat{f}_{\bar{z}}\rangle^c.$$

Now let F be a local adapted frame of f, thus a frame of $\hat{f}$ too. Using identity $\hat{f}^*\beta = Ad(F)\alpha_\mathfrak{p}$, and the complex matrix $A_\mathfrak{p} = [F^{-1}F_z]_\mathfrak{p}$ given by (43), with $B_\mathfrak{p} = \overline{A_\mathfrak{p}}$, we obtain

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$$\frac{1}{2}\|d\hat{f}\|^2 = e^{-2u}\langle \hat{f}_z, \hat{f}_{\bar z}\rangle^c = e^{-2u}\left\langle \hat{f}^*\beta\left(\frac{\partial}{\partial z}\right), \hat{f}^*\beta\left(\frac{\partial}{\partial \bar z}\right)\right\rangle^c = e^{-2u}\langle A_{\mathfrak{p}}, B_{\mathfrak{p}}\rangle^c = -e^{-2u}\frac{1}{2}tr(A_{\mathfrak{p}}.B_{\mathfrak{p}})$$
$$= e^{-2u}\left(e^{2u}\left(1-H^2\right) - e^{-2u}|\xi|^2\right) = 1 - H^2 - e^{-4u}|\xi|^2. \tag{64}$$

If λ_1, λ_2 are the principal curvatures of the immersed surface, it is easily seen that

$$e^{-4u}|\xi|^2 = \frac{1}{4}(\lambda_1 - \lambda_2)^2.$$

Therefore on a relatively compact domain $\Omega \subset M$ we obtain the following formula for the energy of $\hat{f}$,

$$E_\Omega(\hat{f}) = \int_\Omega \left[1 - H^2 - \frac{1}{4}(\lambda_1 - \lambda_2)^2\right] dA, \tag{65}$$

where dA is the area element of (M, g).

On the other hand if M is compact without boundary, the Willmore energy of the conformal immersion $f : M \to \mathbb{S}^3_1$ is given by

$$W(f) = \int_M \left(K + H^2 - 1\right) dA = \frac{1}{4}\int_M (\lambda_1 - \lambda_2)^2\, dA. \tag{66}$$

Combining (65) and (66) with Gauss equation (7) and the Gauss–Bonnet formula we obtain

Lemma 7.1. *Let $f : M \to \mathbb{S}^3_1$ be a conformal immersion of a compact closed Riemann surface M. Then the total energy of $\hat{f}$ over M and the Willmore energy $W(f)$ are related by the equality*

$$2W(f) = 2\pi\mathcal{X}(M) - E(\hat{f}), \tag{67}$$

where H and K are the mean curvature and the Gaussian curvature respectively of the immersed surface, and $\mathcal{X}(M)$ is the Euler–Poincaré characteristic of M.

Since $W(f) \geqslant 0$ for every conformal immersion f, we deduce that

$$E(\hat{f}) \leqslant 2\pi\mathcal{X}(M). \tag{68}$$

If $f : M \to \mathbb{S}^3_1$ has constant mean curvature H satisfying $H^2 < 1$, Akutagawa [1] and independently Ramanathan [13] proved that $f(M)$ is a totally umbilic 2-sphere with constant Gaussian curvature $K = 1 - H^2 > 0$. Thus $W(f) = 0$ and so equality is attained in (68), namely $E(\hat{f}) = 4\pi$. Thus there are no compact genus zero surfaces which are vacua of the twistor energy.

On the other hand from (67) since $\mathcal{X}(T^2) = 0$, we obtain $E(\hat{f}) \leqslant 0$ for every conformal immersion $f : T^2 \to \mathbb{S}^3_1$ of the two torus $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$. Thus the mean curvature function H of these immersions satisfies $H^2 \geqslant 1$.

One may wonder if there are spacelike tori with zero twistor energy. Assume that $f : T^2 \to \mathbb{S}^3_1$ is a conformal immersion with $E(\hat{f}) = 0$, then its mean curvature should satisfy $H^2 = 1$. Lifting f to the universal covering $\mathbb{R}^2 \to T^2 = \mathbb{R}^2/\Gamma$ we obtain a double periodic (with respect to Γ) conformal immersion $\tilde{f} : \mathbb{R}^2 \to \mathbb{S}^3_1$ such that $\tilde{f}(\mathbb{R}^2) = f(T^2)$. The corresponding complex holomorphic function $\tilde{\xi}$ is entire and double-periodic, hence constant. If f is umbilic free, then $\tilde{\xi}$ never vanishes and one can normalize (by a change of coordinate) so that $\tilde{\xi} = 1$ on $\mathbb{R}^2$ and so $\xi = 1$ on T^2. Thus the conformal immersion f is free of umbilic points on the whole T^2, so that its principal curvatures never coincide on T^2. In particular Eq. (7) becomes

$$K = \frac{1}{4}(\lambda_1 - \lambda_2)^2 > 0$$

on the whole T^2. Integrating this equation we have

$$0 = \mathcal{X}\left(T^2\right) = \int_{T^2} K\, dA = W(f) > 0.$$

This shows that there is no genus one umbilic free spacelike surface in $\mathbb{S}^3_1$ with zero-energy twistor lift. On the other hand it is not difficult to rule out the existence of totally umbilic spacelike tori with $H^2 = 1$, i.e. vacua of the twistor energy. Thus the search of genus $g > 1$ compact spacelike vacua of the twistor energy seems to be an interesting problem.

Remark 7.1. Conformally immersed umbilic free $f : T^2 \to \mathbb{S}^3_1$ with constant mean curvature H satisfying $H^2 > 1$ are determined by double periodic solutions of Gauss equation (4) which normalized is the Sinh–Gordon equation:

$$u_{\bar{z}z} + \sin h(u) = 0,$$

see [2,3,12].

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